Prof. Dr. H. D. Alber

WS 07/08
30. 11. 07

AG 06, FB Mathematik
Tech. Univ. Darmstadt

Partial Differential Equations I: Linear Theory
Tutorial 07: Exercises[1]

In this tutorial we are going to investigate properties of the solutions to the Laplace equations in $\Omega \subset \mathbb{R}^2$, related to the mean value formula. As an application of such formula we prove the Liouville theorem. Note that most of the properties discussed in this tutorial are still valid in higher dimensions.

1. Prove that the "circle" mean value formula implies the "disk" mean value formula. Namely, for the solution u to the equation $\Delta u = 0$ in Ω, there holds

$$u(x) = \frac{1}{2\pi R} \int_{|x-y|=R} u(y) dS_y,$$

then we can infer from the above equality that

$$u(x) = \frac{1}{\pi R^2} \int_{|x-y|\leq R} u(y) dy,$$

Remark: In general case, the words "circle, disk" are replaced by "sphere, ball", respectively. □

2. Suppose that u is a solution to the Laplace equation $\Delta u = 0$ in $B_R(0)$. Let r be a real number such that $0 < r < R$.

a) Make use of the Green Formula to prove that

$$\int_0^{2\pi} \frac{\partial u}{\partial r}(x_1 + r\cos\theta, x_2 + r\sin\theta) d\theta = 0. \tag{1}$$

Here $\frac{\partial u}{\partial r}$ denotes the radial derivative of u.

[1] If you have any opinion and/or suggestion on the Tutorial, please send your email to Prof. Dr. H.D. Alber at alber@mathematik.tu-darmstadt.de, or to Dr. P. Zhu at zhu@mathematik.tu-darmstadt.de.

b) Integrate (1) with respect to r over $(0, R)$ and use the Laplace equation to prove the mean value formula

$$u(x) = \frac{1}{2\pi R} \int_{|x-y|=R} u(y) dS_y.$$

The following problem is your homework.

3. (**The Liouville Theorem**) Suppose that $u : \mathbb{R}^2 \to \mathbb{R}$ satisfies $\Delta u = 0$ in $\mathbb{R}^2$ and is bounded. Show that $u(x) \equiv$ Const.
(**Hints:** Invoking the "disk" mean value formula in Problem 1, we have for any $r > 0$ and $x_1, x_2 \in \mathbb{R}^2$ that

$$u(x_1) - u(x_2) = \frac{1}{\pi r^2} \left(\int_{|x_1-y|\le r} u(y) dy - \int_{|x_2-y|\le r} u(y) dy \right),$$

we rewrite the right-hand side as

$$\frac{1}{\pi r^2} \left(\int_{B_r(x_1)\setminus B_r(x_2)} u(y) dy - \int_{B_r(x_2)\setminus B_r(x_1)} u(y) dy \right).$$

Then estimate the quantity of $\frac{1}{\pi r^2} Vol(B_r(x_1)\setminus B_r(x_2))$ (which converges to 0 as $r \to \infty$). Here Vol denotes the volume, and for $i = 1, 2$, $B_r(x_i)$ are balls in $\mathbb{R}^2$ centered at x_i with radius r. *Another way* is to consider the two balls with different radii R, r (say $R = |x_1 x_2| + r$) so that $B_r(x_2) \subset B_R(x_1)$.)