Prof. Dr. H. D. Alber

WS 07/08
23. 11. 07

AG 06, FB Mathematik
Tech. Univ. Darmstadt

# Partial Differential Equations I: Linear Theory

## Tutorial 06: Exercises[1]

This tutorial is concerned with the maximum principle for elliptic equations.

**1.** If the condition $g(x) \geq 0$ (see the first theorem in Section 5.1 of the lecture notes) is not satisfied, then the maximum principle is not valid. Consider the following example:

$$\begin{cases} u''(x) + \pi^2 u(x) = 0, & x \in (0,1), \\ u(0) = u(1) = 0. \end{cases}$$

**2.** Let $\Omega \subset \mathbb{R}^n$ with $n \in \mathbb{N}$ be a bounded open set and $f : \Omega \to \mathbb{R}$ be continuous. Suppose that $u$ is a classical solution to the nonlinear equation

$$\Delta u - u^3 = f, \text{ in } \Omega. \tag{1}$$

a) Show that for all $x \in \Omega$ the following assertions are true

$$u(x) \leq \max\left(0, \max_{y \in \partial\Omega} u(y)\right), \text{ if } f(x) \geq 0 \text{ in } \Omega,$$

$$u(x) \geq \min\left(0, \min_{y \in \partial\Omega} u(y)\right), \text{ if } f(x) \leq 0 \text{ in } \Omega.$$

b) Assume further that $u$ satisfies the Dirichlet boundary condition

$$u|_{\partial\Omega} = u^{(b)},$$

where $u^{(b)} \in C(\partial\Omega)$, and that $v$ is another solution to (1) satisfying the same boundary condition

$$v|_{\partial\Omega} = u^{(b)}.$$

[1] If you have any opinion and/or suggestion on the Tutorial, please send your email to Prof. Dr. H.D. Alber at alber@mathematik.tu-darmstadt.de, or to Dr. P. Zhu at zhu@mathematik.tu-darmstadt.de.

Prove that

$$u \equiv v \text{ in } \bar{\Omega},$$

so the solution to the Dirichlet boundary value problem of (1) is unique.

---

The following problem is your homework.

**3.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and $f, a_i : \Omega \to \mathbb{R}$, $i = 1, \cdots, n$, be continuous in $\bar{\Omega}$. Suppose that $u \in C(\bar{\Omega}, \mathbb{R})$ is two times differentiable in $\Omega$ and satisfies

$$\Delta u(x) + \sum_{k=1}^{n} a_i(x) \frac{\partial}{\partial x_i} u(x) - g(x)u(x) = f(x), \ x \in \Omega.$$

Prove the following are true:

i) If $g(x) \geq 0$ in $\Omega$, then for all $x \in \Omega$

$$u(x) \leq \max\left(0, \max_{y \in \partial\Omega} u(y)\right), \text{ if } f(x) \geq 0 \text{ in } \Omega,$$

$$u(x) \geq \min\left(0, \min_{y \in \partial\Omega} u(y)\right), \text{ if } f(x) \leq 0 \text{ in } \Omega.$$

ii) If $g(x) > 0$ in $\Omega$, then for all $x \in \Omega$

$$u(x) \leq 0, \text{ or } u(x) < \max_{y \in \partial\Omega} u(y), \text{ if } f(x) \geq 0 \text{ in } \Omega,$$

$$u(x) \geq 0, \text{ or } u(x) > \min_{y \in \partial\Omega} u(y), \text{ if } f(x) \leq 0 \text{ in } \Omega.$$