Prof. Dr. H. D. Alber

WS 07/08
16. 11. 07

AG 06, FB Mathematik
Tech. Univ. Darmstadt

Partial Differential Equations I: Linear Theory
Tutorial 05: Exercises[1]

In this tutorial, we are going to investigate the properties of the Bessel functions and to rewrite the formula in Theorem 4.2 as Possion formula for an analytic function, which is equivalent to the Cauchy integral formula. Also we study a function which is infinitely continuously differentiable and will be applied to the density theorems later.

1. a) Remember that the Bessel function is defined by

$$\mathcal{J}_m(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!} \left(\frac{z}{2}\right)^{2k+m}, \quad m \in \mathbb{N}_0.$$

Show that

$$\frac{d}{dz}\{z^m \mathcal{J}_m(z)\} = z^m \mathcal{J}_{m-1}(z).$$

b) Prove the recurrence formulae:

$$\mathcal{J}_{m-1}(z) + \mathcal{J}_{m+1}(z) = \frac{2m}{z}\mathcal{J}_m(z), \qquad \mathcal{J}_{m-1}(z) - \mathcal{J}_{m+1}(z) = 2\mathcal{J}_m'(z),$$

$$z\mathcal{J}_m'(z) + m\mathcal{J}_m(z) = z\mathcal{J}_{m-1}(z), \qquad z\mathcal{J}_m'(z) - m\mathcal{J}_m(z) = -z\mathcal{J}_{m+1}(z).$$

c) (**Orthogonality relation**) We denote the zeros of $\mathcal{J}_m(z)$ by $k_{m,j}$, $j \in \mathbb{N}_0$. Prove that

$$\int_0^1 r\mathcal{J}_m(k_{m,i}r)\mathcal{J}_m(k_{m,j}r)dr = 0, \quad \text{if} \quad i \neq j.$$

2. (**The Possion formula.**) Invoking Theorem 4.2 in the lecture. Prove that the formula for u in Theorem 4.2 can be rewritten as

$$u(x) = u(r,\varphi) = \frac{R}{2\pi}\int_0^{2\pi} \frac{u^{(b)}(\theta)d\theta}{R - re^{i(\varphi-\theta)}}.$$

[1]If you have any opinion and/or suggestion on the Tutorial, please send your email to Prof. Dr. H.-D. Alber at alber@mathematik.tu-darmstadt.de, or to Dr. P. Zhu at zhu@mathematik.tu-darmstadt.de.

(You may assume that that $a_m = 0$ if $m < 0$, for simplicity.)

Moreover, this formula is equivalent to the Cauchy integral formula i.e.

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w - z} dw,$$

here, Γ is chosen to be a special path $\Gamma = \{w \in \mathbb{R}^2 \mid |w| = R\}$.

The following problem is your homework.

3. Show that the function

$$g(t) := \begin{cases} C \exp\left(\frac{-1}{t}\right), & \text{for all } t > 0, \\ 0, & \text{for all } t < 0 \end{cases}$$

is infinitely continuously differentiable on $\mathbb{R}$. Here C is a positive constant which is usually chosen so that $\int_{\mathbb{R}} g(t)dt = 1$.

Remark: This function plays an important role in the theory of Sobolev spaces which are crucial in the theory of PDEs, we shall make use of it very often later on.

(**Hint:** We need only to investigate differentiability of $g(t)$ at the point $t = 0$. The n-th order derivative of $\exp\left(\frac{-1}{t}\right)$ can be written as $P_n(\frac{1}{t}) \exp\left(\frac{-1}{t}\right)$ with $P_n(t)$ being a polynomial of t, for $t > 0$.)