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WS 07/08
09. 11. 07

AG 06, FB Mathematik
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Partial Differential Equations I: Linear Theory
Tutorial 04: Exercises[1]

This tutorial is concerned with the Leibnitz formula for weak derivatives, some properties of the Sobolev spaces, and with a Neumann boundary value problem for the Helmholtz equation. As an application of Hölder's inequality we prove an interpolation inequality as a homework problem. Let $n \in \mathbb{N}_0$, let Ω be an open bounded set in $\mathbb{R}^n$ with smooth boundary.

1. Let α, β be two multi-indices. We say $\alpha \geq \beta$ provided $\alpha_i \geq \beta_i$ for all $i = 1, 2, \cdots, n$. We denote $\alpha! = \alpha_1! \cdot \alpha_2! \cdots \alpha_n!$ and if $\alpha \geq \beta$,

$$\binom{\alpha}{\beta} = \frac{\alpha!}{(\alpha-\beta)!\,\beta!} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n}.$$

Let $m \in \mathbb{N}$. Assume that $u \in H_m(\Omega)$, $\varphi \in C_\infty^*(\Omega)$. Prove for all α with $|\alpha| \leq m$, the Leibniz formula holds

$$D^\alpha(u \cdot \varphi) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta u \cdot D^{\alpha-\beta}\varphi.$$

2. a) Let X be a Banach space. We call X a *Banach algebra*, if for any u, v of X, their product $u \cdot v$ is also an element of X.

Let (a, b) be an interval, where $a < b$. Prove that $H_1((a, b))$ is a Banach algebra, that is for $u, v \in H_1((a, b))$ there holds

$$u \cdot v \in H_1((a, b)).$$

(**Hint.** Use the Leibnitz formula in problem 1 and an inequality proved in the lecture, i.e. $|u(x)| \leq r^{\frac{1}{2}} \|u'\|_{(a,b)} + r^{-\frac{1}{2}} \|u\|_{(a,b)}$ for all $0 < r \leq b - a$. This result

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can be generalized. Namely, $H_m(\Omega)$ is a Banach algebra for $m > \frac{n}{2}$, where n is the dimension of domain Ω.)

b) Let $\alpha, \beta \in (0,1)$. Suppose that $u \in C_\alpha(\Omega)$, $v \in C_\beta(\Omega)$ and that $F : \mathbb{R} \to \mathbb{R}$ is a Hölder continuous function with exponent α. Show that

i) $C_\beta(\Omega) \subset C_\alpha(\Omega)$, if $\beta \geq \alpha$,
ii) $u \cdot v \in C_\gamma(\Omega)$, where $\gamma = \min\{\alpha, \beta\}$,
iii) $F(v) \in C_\gamma(\Omega)$, where $\gamma = \alpha \cdot \beta$.

3. Consider the Neumann boundary value problem for the Helmholtz equation

$$\begin{aligned} \Delta u + \lambda u &= f, \text{ in } \Omega, \\ \frac{\partial u}{\partial n} &= 0, \text{ on } \partial\Omega. \end{aligned}$$

Here n is the outward normal vector, λ is a given number and $f \in L^2(\Omega)$ is a given function.

Define weak solutions for this problem, and show that a weak solution $u \in C_2^*(\Omega) \cap C_1(\bar{\Omega})$ is a classical solution to the above problem.

The following problem is your homework.

4. (**An application of the Hölder inequality**) Let $1 \leq p < q < r$, such that

$$\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{r}$$

for some $\theta \in (0,1)$. If $u \in L_p(\Omega) \cap L_r(\Omega)$, then $u \in L_q(\Omega)$ and

$$\|u\|_{L_q(\Omega)} \leq \|u\|^\theta_{L_p(\Omega)} \|u\|^{1-\theta}_{L_r(\Omega)}.$$

(**Hint.** To apply the Hölder inequality, we write $|u|^q = |u|^{q\theta} \cdot |u|^{q(1-\theta)}$, let $d = \frac{p}{q\theta}$ and $d' = \frac{r}{q(1-\theta)}$, then get $\frac{1}{d} + \frac{1}{d'} = 1$.)