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AG 06, FB Mathematik
Tech. Univ. Darmstadt

Partial Differential Equations I: Linear Theory
Tutorial 03: Exercises[1]

In the present tutorial, we are going to study an eigenvalue problem with Neumann boundary conditions, the regularity of solution to an boundary value problem and investigate some applications of the projection theorem. Let $\Omega = (a, b)$ with $a, b \in \mathbb{R}$ in problems 1, 2 or let Ω be an open subset in $\mathbb{R}^2$ in problem 3. Let λ be a real number.

1. Solve the following eigenvalue problem:

$$\begin{aligned} u'' + \lambda u &= 0, \text{ in } \Omega, \\ u'(a) &= u'(b) = 0. \end{aligned}$$

Here u is an unknown. Does there exist a complete orthonormal system of eigenfunctions? Note the difference between this problem and the eigenvalue problem with Dirichlet boundary conditions.

2. (**Regularity**) Let $f \in C_m(\bar{\Omega})$. Suppose that $u \in C_2(\Omega) \cap C(\bar{\Omega})$ is a solution to the following problem

$$\begin{aligned} u'' + \lambda u &= f, \text{ in } \Omega, \\ u(a) &= u(b) = 0. \end{aligned}$$

Prove that $u \in C_{m+2}(\bar{\Omega})$.

3. Let f, g be two functions defined in $\Omega \subset \mathbb{R}^2$, with L^2-weak derivative $D^\alpha f, D^\alpha g$ where $|\alpha| \leq 2$. Prove for any number c that the following formulas are true:

$$\begin{aligned} D_x(cf) &= c\, D_x f, \\ D_x(f+g) &= D_x f + D_x g, \\ D_x D_y f &= D_y D_x f. \end{aligned}$$

[1]If you have any opinion and/or suggestion on the Tutorial, please send your email to Prof. Dr. H. D. Alber at alber@mathematik.tu-darmstadt.de, or to Dr. P. Zhu at zhu@mathematik.tu-darmstadt.de.

4. (**Orthonormal system or ON-system**) A family of functions $(f_\alpha)_{\alpha\in A}$ (A is a countable set) is called an orthonormal system (ON-system) if

$$(f_\alpha, f_\beta) = \begin{cases} 0, & \text{if } \alpha \neq \beta; \\ 1, & \text{if } \alpha = \beta. \end{cases}$$

For a function f, we define $c_\alpha = (f, f_\alpha)$, i.e. the Fourier coefficients of f with respect to the ON-system $(f_\alpha)_{\alpha\in A}$.

Prove that if E is a finite subset of A, and $\sum_{\alpha\in E} u_\alpha f_\alpha$ is a linear combination of f_α with complex coefficients u_α, then the L^2-norm of

$$f - \sum_{\alpha\in E} u_\alpha f_\alpha$$

attains its minimum if there holds $u_\alpha = c_\alpha$ for all $\alpha \in E$.
(**Hint:** Consider $\mathrm{span}\{f_\alpha\}_{\alpha\in E}$ and use the projection theorem.)

The following problem is your homework.

5. Let X be a Hilbert space, let $Y \subsetneqq X$ be a closed subspace. Then there is a closed subspace $Y^\perp$, such that

$$(u, v) = 0,$$

for all $u \in Y,\ v \in Y^\perp$, and such that $Y + Y^\perp = X$. Every $x \in X$ can be written in a unique way as

$$x = y_1 + y_2,\ \text{with } y_1 \in Y,\ y_2 \in Y^\perp.$$