Prof. Dr. H. D. Alber

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Tech. Univ. Darmstadt

Partial Differential Equations I: Linear Theory
Tutorial 02: Exercises[1]

Beginning with some exercises for multi-indices, we will in this tutorial prove the Duhamel Principle, the Young inequality and the Hölder inequality, also we are going to discuss an application of the Projection Theorem in Exercise 2. Let us first introduce some

Notations: Let $\alpha, \alpha_i \in \mathbb{N}_0^n$ be multi-indices, here $i = 0, 1, \cdots, n$ and $n \in \mathbb{N}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $\alpha = (\alpha^1, \cdots, \alpha^n)$ with $\alpha^j \in \mathbb{N}_0, j = 1, \cdots, n$, we define for $x \in \mathbb{R}^n$, $D^\alpha := \partial_{x_1}^{\alpha^1} \cdots \partial_{x_n}^{\alpha^n}$.

$C^3(\mathbb{R}^2)$ is the set of real functions that have continuous derivatives up to third order.

1. Assume that $u = u(x)$ is a real function in $C^3(\mathbb{R}^2)$ and $c_\alpha(x)$ are the coefficient functions. By introducing the multi-index, we can simplify considerably the writing of an operator. For instance, the first order linear operator $\sum_{|\alpha| \le 1} c_\alpha(x) D^\alpha u(x)$ includes many single terms like $c_{\alpha_i}(x) \frac{\partial u}{\partial x_i}$ $(i = 1, 2, \cdots, n)$ and $c_{\alpha_0}(x)u$. Here, the i-th component of α_i is 1 and the others are 0, while all components of α_0 are 0. More precisely, we have

$$\sum_{|\alpha| \le 1} c_\alpha(x) D^\alpha u(x) = c_{\alpha_0}(x) u + \sum_{i=1}^n c_{\alpha_i}(x) \frac{\partial u}{\partial x_i}$$

a) Do not use the multi-index notation and rewrite the following operators

$$\sum_{|\alpha| \le 2} c_\alpha(x) D^\alpha u(x), \qquad \sum_{|\alpha| = 3} c_\alpha(x) D^\alpha u(x)$$

as done in the above example.

[1] If you have any opinion and/or suggestion on the Tutorial, please send your email to Prof. Dr. H. D. Alber at alber@mathematik.tu-darmstadt.de, or to Dr. P. Zhu at zhu@mathematik.tu-darmstadt.de.

b) Rewrite the following equations

$$u_{tt} - c^2 u_{xx} = 0, \text{ in } \mathbb{R}^+ \times \mathbb{R}; \quad \Delta u = 0, \text{ in } \mathbb{R}^2$$

in the multi-index form and calculate the coefficient functions.

2. (**Duhamel Principle**) Consider the initial boundary value problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), \; x \in (0, \ell), \; t \in (0, \infty), \\ u|_{t=0} = 0, \quad u_t|_{t=0} = 0, \\ u|_{x=0} = 0, \quad u|_{x=\ell} = 0. \end{cases} \tag{1}$$

Let $w = w(x,t;\tau)$ be a solution to the following problem

$$\begin{cases} w_{tt} - c^2 w_{xx} = 0, \quad x \in (0,\ell), \quad t \in (\tau, \infty), \\ w|_{t=\tau} = 0, \quad w_t|_{t=\tau} = f(x,\tau), \\ w|_{x=0} = 0, \quad w|_{x=\ell} = 0. \end{cases}$$

Prove that

$$u(x,t) = \int_0^t w(x,t;\tau)d\tau$$

is a solution to Problem (1).

3. Let X be a Hilbert space over $\mathbb{C}$ and let Y be a closed subspace of X. Let $z \in X \setminus Y$.

Find a bounded linear functional $f : X \to \mathbb{C}$ such that

$$\begin{aligned} f(x) &= 0, \text{ for all } x \in Y, \\ f(z) &= 1. \end{aligned}$$

(**Hint:** Use the projection theorem.)

The following problem is your homework.

4. Suppose that $a, b \geq 0$ are real numbers, f, g are real functions defined in the domain Ω. Prove

a) the **Young** inequality

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q, \quad \text{and} \quad ab \leq \frac{1}{p}\varepsilon^p a^p + \frac{1}{q}\varepsilon^{-q}b^q$$

for any $1 \leq p, q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Here, p is called the **conjugate exponent** to q, and *vice versa*. And ε is an arbitrary positive number.

b) and the **Hölder** inequality

$$\left|\int_\Omega f(x)g(x)dx\right| \le \left(\int_\Omega |f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_\Omega |g(x)|^q dx\right)^{\frac{1}{q}}.$$

where $|f|^p, |g|^q$ are integrable.

(**Hints:** *Approach 1.* Apply the property that the exponent function $f = \exp(x)$ is a convex function, i.e. $f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y),\ \forall \lambda \in [0,1]$. On the other hand, we can write $ab = \exp(\log a + \log b) = \exp\left(\frac{1}{p}\log(a^p) + \frac{1}{q}\log(b^q)\right)$.

Approach 2. Regard ab as the area of certain rectangle, then compare it with the sum of the areas of the regions confined, respectively, by the graph of the function $f = x^{p-1}$(on $[0,a]$) and the x-axis (from 0 to a), the graph of $g = y^{q-1}$(on $[0,b]$) and the y-axis (from 0 to b).)

Remark: For the limit case $p = 1, q = \infty$ (or $q = 1, p = \infty$) the Young inequality should be understood in the following way: the limit as $p \downarrow 1$ (which implies $q \uparrow \infty$) satisfies the Young inequality. For $0 \le b \le 1$, there holds $\lim_{p\downarrow 1}\left(\frac{1}{p}a^p + \frac{1}{q}b^q\right) = a$; If $b > 1$, we have $\lim_{p\downarrow 1}\left(\frac{1}{p}a^p + \frac{1}{q}b^q\right) = \infty$. Therefore, it is easy to see that for both cases the Young inequality is valid. □