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AG 06, FB Mathematik
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Partial Differential Equations I: Linear Theory
Tutorial 1: Exercises[1]

In this tutorial, I am going to present you some basic knowledge about partial differential equations, then give some exercises. In what follows, we assume that Ω is an open bounded interval, $\Omega = (a, b)$, and we denote its closure by $\overline{\Omega}$. Here a, b are two real numbers.

1. a) Suppose that $f = f(x)$ is a real continuous function defined on $\overline{\Omega}$, which is given and satisfies

$$\int_\Omega f(x)\varphi(x)dx = 0 \tag{1}$$

for all real functions $\varphi : \overline{\Omega} \to \mathbb{R}$ with $\varphi(a) = \varphi(b) = 0$.

Then we have

$$f(x) = 0, \quad \forall\, x \in \overline{\Omega}.$$

Prove the above assertion.

b) If we assume weaker condition: f is continuous in Ω, what can we conclude?

2. (The **d'Alembert** formula) Consider the following initial value problem (IVP)

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0, \text{ for all } (x,t) \in \mathbb{R} \times (0, +\infty), \\ u|_{t=0} &= u_0, \\ u_t|_{t=0} &= u_1, \end{aligned}$$

where c is a positive constant and u_0, u_1 are two given continuous functions.

Suppose that the first and second order derivatives of u_0 and the first order derivative of u_1 are continuous. Prove that

$$u(x,t) = \frac{1}{2}\left(u_0(x+ct) + u_0(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(\xi)d\xi$$

[1]If you have any opinion and/or suggestion on the Tutorial, please send your email to Prof. Dr. H. D. Alber at alber@mathematik.tu-darmstadt.de, or to Dr. P. Zhu at zhu@mathematik.tu-darmstadt.de.

is a solution to the above IVP.

Hint. Consider the ansatz $u(x,t) = f(x+ct) + g(x-ct)$.

The following problem is your homework.

3. (Decomposition of the **d'Alembert** operator) Suppose that $u = u(x,t)$ is a twice continuously differentiable function. Define $\Box u = u_{tt} - c^2 u_{xx}$ ($\Box$ is called the **d'Alembert** operator), $D_+ u = u_t + c u_x$ and $D_- u = u_t - c u_x$. Then the second order wave equation can be rewritten as

$$\Box u = 0. \tag{2}$$

Prove that (2) can be "factored" into

$$D_+ D_- u = 0, \quad \text{or} \quad D_- D_+ u = 0,$$

moreover, if we introduce a new function $v = D_- u$, then the scalar equation (2) is equivalent to the following system of two first order equations:

$$\begin{cases} D_+ v = 0, & \textit{i.e.}\ v_t + c v_x = 0, \\ D_- u = v, & \textit{i.e.}\ u_t - c u_x = v. \end{cases}$$