TU Darmstadt
Fachbereich Mathematik
Wilhelm Stannat

WS 2007/08
23.01.08

Probability Theory
14. Aufgabenblatt

Gruppenübungen

Aufgabe G40:

The conditional variance is defined as

$$Var[X|Y] := E[(X - E[X|Y])^2|Y].$$

Prove (and interpret) the following formula for the total variance,

$$Var[X] = E[Var[X|Y] + Var[E[X|Y]].$$

Aufgabe G41:

Let $Y_1, \ldots, Y_{n+1}$ be random variables. Prove that for every $X \in \mathcal{L}^2$

$$E\left[(E[X|Y_1, \ldots, Y_{n+1}] - X)^2\right] \le E\left[(E[X|Y_1, \ldots, Y_n] - X)^2\right],$$

and interpret this in terms of prediction.

Aufgabe G42:

Consider a prediction problem in which the set of allowable predictors is $V = \{aY + b : a, b \in \mathbb{R}\}$, where $Y \in \mathcal{L}^2$ and $\sigma^2 = Var[Y] > 0$.

(a) Show that $Z_1 = (Y - E[Y])/\sigma$ and $Z_2 \equiv 1$ constitute an orthonormal basis of V.

(b) Show that for $X \in \mathcal{L}^2$

$$\widehat{X} = \frac{Cov(X,Y)}{\sigma^2}(Y - E[Y]) + E[X]$$

is the orthogonal projection in $\mathcal{L}^2$ of X onto V.

Aufgabe G43:

Let Ω be a set and $\mathcal{A}$ its σ-algebra. Let μ and ν be measures on $\mathcal{A}$ with $\nu \le \mu$ and μ σ-finite (i.e. there exist $A_n \in \mathcal{A}$ with $\mu(A_n) < \infty$ and $\cup_{n=1}^{\infty} A_n = \Omega$). Prove the existence of an $\mathcal{A}$-measurable function f with values $0 \le f \le 1$ and $\nu = f\mu$.

Hausübungen

Aufgabe H39:**

A random vector $X = (X_1, \dots, X_n)$ has a multivariate normal distribution with mean vector m and covariance matrix $Q = CC^T$, if $X = CY + m$, where $Y = (Y_1, \dots, Y_n)$ and $Y_1, \dots, Y_n$ are $N(0,1)$-distributed.

(i) Suppose that (X_1, X_2) has a multivariate normal distribution. Show that the following are equivalent:

(a) X, Y are independent.

(b) X, Y are uncorrelated.

(ii) Suppose that (X_1, X_2) has a multivariate normal distribution. Show that the orthogonal projection $\widehat{X}_1$ of X_1 onto the subspace $V = \{aX_2 + b : a, b \in \mathbb{R}\}$ satisfies the definition of $E[X_1|X_2]$. In this case, prediction and linear prediction are synonymous.

Aufgabe H40*:**

Let X, W be two independent multivariate normal-distributed random variables with mean vectors $m_X, m_W = 0$ and covariance matrices Q_X, Q_W. Furthermore let $Y = BX + W$. Show that

$$E[X|Y] = Q_X B^T (BQ_X B^T + Q_W)^{-1}(Y - Bm_X) + m_X.$$

Hint: Show that (X, W) has a multivariate normal distribution, hence (X, Y) has a multivariate normal distribution, too. Now, use the multivariate analogue of H39, (ii).