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Probability Theory
13. Aufgabenblatt

Gruppenübungen

Aufgabe G37:
Let $B_n \in \mathcal{A}$, $n \in \mathbb{N}$, be pairwise disjoint sets with $\Omega = \bigcup_{n\in\mathbb{N}} B_n$ and $\mathcal{A}_0 = \sigma\{B_n | n \in \mathbb{N}\}$. Show that for every random variable $X \geq 0$

$$E[X|\mathcal{A}_0] = \sum_{n\in\mathbb{N}:P[B_n]>0} E[X|B_n]1_{B_n}.$$

Aufgabe G38:
Let $I \subset \mathbb{R}$ be an interval, X be an integrable random variable with values in I and $q : I \to \mathbb{R}$ be a convex function. Show that:

(a) For every sub-σ-algebra $\mathcal{A}_0 \subset \mathcal{A}$ it follows that $E[X|\mathcal{A}_0] \in I$ P-a.s.

(b) If $q(X)$ is integrable, then

$$q(E[X|\mathcal{A}_0]) \leq E[q(X)|\mathcal{A}_0] \qquad \text{P-a.s.}$$

In exercises G39 and H36 let X, Y be random variables with joint density $f(x, y)$. Define

$$f_Y(y) := \int_{\mathbb{R}} f(x, y)\, dx,\ y \in \mathbb{R}$$

and

$$f_{X|Y}(x|y) := \begin{cases} \frac{1}{f_Y(y)} f(x, y) & \text{if } f_Y(y) > 0, \\ 0 & \text{if } f_Y(y) = 0. \end{cases}$$

$f_{X|Y}$ is called the conditional density of X given Y.

Aufgabe G39:
Show that if X and Y are independent, the conditional densitiy $f_{X|Y}(x, y)$ is independent of y and only a function of the variable x.

Hausübungen

Aufgabe H36:

Let $g : \mathbb{R} \to \mathbb{R}_+$ be measurable. Then

$$\int_{\mathbb{R}} g(x) f_{X|Y}(x|Y)\, dx \tag{1}$$

is a version of the conditional expectation $E[g(X)|Y]$.

Aufgabe H37:

Let μ be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $P = \bigotimes_{k=1}^n \mu$ be the n-fold product on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Let $\mathcal{A}$ be the system of all sets $B \in \mathcal{B}(\mathbb{R}^n)$ with the property that for all permutations $i_1, \dots, i_n$ of $1, \dots, n$ the point $(x_{i_1}, \dots, x_{i_n})$ is in B. Show that:

(i) $\mathcal{A}$ is a sub-σ-algebra of $\mathcal{B}(\mathbb{R}^n)$.

(ii) For every integrable random variable X on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), P)$ it follows that

$$\frac{1}{n!} \sum_{(i_1,\dots,i_n)} X(x_{i_1}, \dots, x_{i_n})$$

a version of the conditional expectation $E[X|\mathcal{A}]$.

Aufgabe H38:

Let $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}$ be sub-σ-algebras and $X \in \mathcal{L}^1$, $X \geq 0$. Then the following are equivalent:

(i) $E[X|\sigma(\mathcal{A}_1, \mathcal{A}_2)] = E[X|\mathcal{A}_1]$.

(ii) $E[XY|\mathcal{A}_1] = E[X|\mathcal{A}_1]E[Y|\mathcal{A}_1]$ for all $\sigma(\mathcal{A}_1, \mathcal{A}_2)$-measurable $Y \geq 0$.

(iii) $E[XX_2|\mathcal{A}_1] = E[X|\mathcal{A}_1]E[X_2|\mathcal{A}_1]$ for all $\mathcal{A}_2$-measurable $X_2 \geq 0$.