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**Probability Theory**
**12. Aufgabenblatt**

**Gruppenübungen**

**Aufgabe G34:**

Construct measurable spaces and transition probabilities in the following examples:

(i) A random walker on $\mathbb{Z}$ moves one step to the right with probability $p$ and one step to the left with probability $1-p$.

(ii) The random walker of (i) starts in the set $\{0, \cdots, n\}$. If he reaches the boundary 0 (resp. $n$) his next step will be to the right (resp. to the left).

(iii) The Gothenburg/Los Angeles weather: Every day in the city of Gothenburg is either rain or sunshine. With probability 3/4 the weather on the next day will be the same as today. On the contrary in the city of Los Angeles, the probability that, given a sunny day, the next day will be sunny again is 0.9, whereas the probability that a rainy day is followed by another rainy day is only 0.25.

In exercises G35 and G36 let $S$ be a finite set and $K = (K(x,y))_{x,y\in S}$ be a stochastic matrix on $S$, i.e. $K(x,y) \geq 0$ and $\sum_{y\in S} K(x,y) = 1$ for all $x \in S$. Let $\mu$ be an initial distribution on $S$ and $P$ be the unique probability measure determined in Proposition 3.1. on $(\Omega, \mathcal{A})$, where $\Omega = S^{\mathbb{N}_0}$, $\mathcal{A} = \sigma(X_0, X_1, \ldots)$, and $X_n : \Omega \to S$, $n \in \mathbb{N}_0$, denote the canonical projections.

**Aufgabe G35:**

Let $0 \leq k < l < m$ and $x, y \in S$. Show that

$$P[X_m = x \,|\, X_k = y] = \sum_{z\in S} P[X_m = x \,|\, X_l = z] \cdot P[X_l = z \,|\, X_k = y]. \quad (1)$$

**Aufgabe G36:**

Prove that the distribution of $X_n$ under the measure $P$ is given by

$$P[X_n = x] = \mu \cdot K^n(x), \quad n \geq 0. \quad (2)$$

Here, $K^n$ denotes the n-th power of the matrix $K$, i.e. $K^0 = Id$, $K^1 = K$, $K^{n+1} = K \cdot K^n$, $n \geq 0$, and the measure $\mu$ is identified with a row vector. (2) are called the n-step transition probabilities of the Markov chain $(X_n)_n$.

**Hausübungen**

**Aufgabe H33:**

Let $S = \{1, 2\}$, $K = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$, $\alpha, \beta \in [0, 1)$ with $\alpha + \beta > 0$.

(i) Determine the n-step transition probabilities $\mu_n$ of the Markov chain $(X_n)$ with transition probability $K$ and given initial distribution $(\gamma, 1 - \gamma)$, $\gamma \in [0, 1]$.

(ii) Prove that $\lim_{n\to\infty} \mu_n = \left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}\right) := \mu_\infty$ and that $\mu_\infty$ is an invariant measure for $K$.

**Aufgabe H34:**

Consider a game of piling up small bricks of the same type. If the resulting tower has reached height $n$, the probability that the tower still stands after adding another brick on top is $p_n$. With probability $1 - p_n$, the tower collapses and a new game starts. Let be $p_0 = 1$.

(i) Determine $S$ and the transition probabilities $K$.

(ii) Assume that $p^* := \sum_{k=1}^{\infty} \prod_{l=1}^{k} p_l < \infty$. Show that the probability measure $\pi(x) = \frac{p^*_{x-1}}{2+p^*}$ with $p^*_x = \prod_{l=1}^{x} p_l$, $x \geq 1$, $p^*_0 := 1$, is an invariant measure for $K$.

**Aufgabe H35:**

Consider the random walker of G34 (i) with $p < \frac{1}{2}$. Assume the walker starts in $\{0, 1, 2, \ldots\}$ and as soon as he reaches 0, he stays with probability $1 - p$ at 0, and with probability $p$ he moves one step to the right. Determine the transition probabilities and the invariant measure.