TU Darmstadt
Fachbereich Mathematik
Wilhelm Stannat

WS 2007/08
19.12.07

Probability Theory
11. Aufgabenblatt

Gruppenübungen

Aufgabe G31:

Show, that for $K(x,\cdot) = N(\alpha x, \sigma^2)$ with $\alpha < 1$, $\sigma^2 > 0$ and $x \in \mathbb{R}$, $\mu = N(0, \frac{\sigma^2}{1-\alpha^2})$ is an equilibrium distribution.

Aufgabe G32:

Two urns contain balls. From the totality of all balls one ball is randomly chosen and placed into the other urn.

Describe this random experiment with a transition probability $K(\cdot,\cdot)$ on the set $S := \mathbb{N}_0^2$ of all pairs (k_1, k_2) of non-negative integers.

Aufgabe G33:

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $X : \Omega \to \mathbb{R}_+$ be a random variable. Prove that

$$E(X) = \int_0^\infty P(X \geq t)\, dt. \tag{1}$$

Hint: Apply Fubini's theorem to

$$(P \otimes dt)(\{(\omega, s) \in \Omega \times \mathbb{R}_+ : 0 \leq s \leq X(\omega)\}).$$

As an application: Use (1) to calculate $E(X)$ for X having exponential distribution with parameter λ.

Hausübungen

Aufgabe H30:

Consider n urns each of which contains s black and w white balls. Now from the first urn one ball is randomly chosen and placed into the second urn. Then from the second urn one ball is randomly chosen and places into the third urn and so on until from the $(n-1)$-th urn one ball is randomly chosen and placed into the n-th urn. At last from the n-th urn one ball is randomly chosen. What is the probability that this ball is white?

Aufgabe H31:

Consider the urn model of exercise G32.

(i) Let $\lambda > 0$ and μ be the probability distribution on S with weights

$$\mu(k_1, k_2) := e^{-\lambda}\frac{\lambda^{k_1}}{k_1!}e^{-\lambda}\frac{\lambda^{k_2}}{k_2!}.$$

Let X_i $(i = 1, 2)$ be the number of balls in urn i, so that X_1, X_2 are independent Poisson-distributed random variables with parameter λ with respect to the measure μ. Show that μ is an equilibrium distribution for the dynamic described by $K(\cdot, \cdot)$, i.e. $\mu K = \mu$.

(ii) What can be said, if the number N of all balls is known?

Aufgabe H32 (Borel-paradox):

Let $\Omega = S^2$ be the unit shere on $\mathbb{R}^3$ and $\mathcal{A}$ be the Borel σ-algebra. Furthermore let P be the uniform distribution on $(\Omega, \mathcal{A})$. For a point $p \in \Omega$ let $(\psi(p), \theta(p))$ be its polar coordinates with $\psi(p) \in [-\pi, \pi)$ and $\theta(p) \in [-\frac{\pi}{2}, \frac{\pi}{2})$. Then $p = (\cos\theta(p)\cos\psi(p), \cos\theta(p)\sin\psi(p), \sin\theta(p))$.

(i) Show that the joint distribution of (ψ, θ) is given by

$$P[\psi \in A, \theta \in B] = \frac{1}{4\pi}\int_{-\pi}^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1_A(\psi)1_B(\theta)|\cos\theta|\,d\psi\,d\theta\,, \quad A, B \in \mathcal{B}(\mathbb{R}).$$

(ii) Consider a density f and two random variables X, Y with joint distribution $P[X \in A, Y \in B] = \int\int 1_A(x)1_B(y)f(x,y)\,dx\,dy$. For y_0 with $\int f(x, y_0)\,dx > 0$ the *conditional distribution of X given Y* is defined by

$$P[X \in A \,|\, Y = y_0] := \frac{\int 1_A(x)f(x,y_0)\,dx}{\int f(x,y_0)\,dx}\,, \quad A \in \mathcal{B}(\mathbb{R}).$$

Show that the conditional distribution of ψ given $\theta = \theta_0$ is the uniform distribution on $[-\pi, \pi)$, but otherwise the conditional distribution of θ given $\psi = \psi_0$ is not the uniform distribution on $[-\frac{\pi}{2}, \frac{\pi}{2})$.