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Probability Theory
10. Aufgabenblatt

Gruppenübungen

Aufgabe G28:

Let X be a geometric distributed random variable with parameter p, $0 < p \leq 1$, i.e. $P(X = k) = p(1-p)^{k-1}$, $k \in \mathbb{N}$. Show that X is memoryless:

$$P(X > m + n \,|\, X > n) = P(X > m) \quad \text{for all } n \in \mathbb{N}.$$

Aufgabe G29 (Polya's urn model):

An urn contains s black and w white balls. In every timestep a ball is randomly choosen and replaced by t balls of the same colour. Let $X_n = 1$, if at the n-th timestep a black ball is taken and otherwise $X_n = 0$. Let P the resulting probability distribution on $\Omega = \{0,1\}^{\mathbb{N}}$. Show that

$$P[X_{n+1} = 1 \,|\, X_1 = x_1, \ldots, X_n = x_n] = \frac{s + (t-1)\sum_{i=1}^n x_i}{s + w + (t-1)n}.$$

Aufgabe G30:

Let Ω be the space of all binary sequences $\omega = (x_1, x_2, \ldots)$, $X_n(\omega) = x_n$ and P_p be the probability distribution of the infinite coin tossing model with parameter $p \in [0,1]$ on Ω. Furthermore let μ be the beta-distribution with parameters $\alpha := \frac{s}{t-1}$ and $\beta := \frac{w}{t-1}$. So μ is a probability distribution on $[0,1]$ with density $\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$. Show for the probability distribution

$$Q[A] := \int_0^1 P_p[A]\, \mu(dp)$$

on Ω that

$$Q[X_{n+1} = 1 \,|\, X_1 = x_1, \ldots, X_n = x_n] = \frac{s + (t-1)\sum_{i=1}^n x_i}{s + w + (t-1)n}.$$

Conclude that Q is the same probability measure as P from exercise G29.

Hint: If μ is beta-distributed with parameters α, β, then

$$\int_0^1 x^n (1-x)^m \, d\mu(x) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \cdot \frac{\Gamma(\beta+m)}{\Gamma(\beta)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+n+m)}.$$

Hausübungen

Bonusaufgabe:

The number of passengers which show up to a flight is often smaller than the number of sold tickets. So the flight companies are selling more tickets than seats available with the risk to pay compensation to overbooked passengers. Suppose the flight company collects $a = 300$ Euro for every passenger who attends to the flight, but looses $b = 500$ Euro for every overbooked passenger. Furthermore assume that every person booking a flight actually shows up with probability $p = 0.95$. How many seats of an

(i) airbus 319 with $S = 124$ seats,

(ii) airbus 380 with $S = 555$ seats

would you sell to maximize the expected gain?

First show that: If $(X_n)_{n\geq 1}$ is a Bernoulli sequence for p, $S_N = \sum_{k=1}^{N} X_k$ and G_N is the gain for N sold tickets, then

$$G_{N+1} - G_N = a\,1_{\{S_N<S\}} X_{N+1} - b\,1_{\{S_N\geq S\}} X_{N+1}.$$

Conclude that $E(G_{N+1}) \geq E(G_N)$ iff $P(S_N < S) \geq b/(a+b)$, and then use normal approximation.

Aufgabe H28:

Prove the following:

(i) Let X be an exponential distributed random variable, then

$$P[X > s + t \,|\, X > t] = P[X > s] \tag{1}$$

for all $s, t \geq 0$. That means X is memoryless.

(ii) Conversely if X is a random variable with $P[X \in (0, \infty)] = 1$ and (1), then X has exponential distribution.

Hint for (ii): Let $\varphi(t) := P[X > t]$. Then φ is decreasing, right continuous and satisfies $\varphi(s + t) = \varphi(s) \cdot \varphi(t)$ for all $s, t \geq 0$.

Aufgabe H29:

Consider the urn model of exercise G29. Prove that the rate of black balls converges weakly to the beta-distribution μ of exercise G30.

Hint: It is known from exercise G29, that

$$P[A] = \int_0^1 P_p[A]\,\mu(dp).$$

The law of large numbers gives $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} X_k = p$ P_p-a.s.