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Fachbereich Mathematik
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Probability Theory
9. Aufgabenblatt

Gruppenübungen

Aufgabe G26:

Use the central limit theorem to prove that

$$\lim_{n\to\infty} e^{-n} \sum_{k=1}^{n} \frac{n^k}{k!} = \frac{1}{2}$$

and

$$\lim_{n\to\infty} \frac{1}{(n-1)!} \int_0^n t^{n-1} e^{-t}\, dt = \frac{1}{2}.$$

Aufgabe G27:

For $n \in \mathbb{N}$ let X_n be a Poisson-distributed random variable with parameter n.

(i) For $a < b$ show that

$$\lim_{n\to\infty} P[n + a\sqrt{n} < X_n < n + b\sqrt{n}] = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2}\, dx.$$

(ii) The number of students in a lecture is Poisson-distributed with parameter $\lambda = 100$. Use the normal approximation (i) to calculate the probability that a lecture hall with 120 seats is sufficient.

Hausübungen

Aufgabe H25:

Let $n \in \mathbb{N}$ and $Y_{n,1}, \ldots, Y_{n,n}$ be a sequence of independent random variables with $P[Y_{n,k} = 1] = 1 - P[Y_{n,k} = 0] = p_{n,k}$ and $S_n := \sum_{k=1}^{n} Y_{n,k}$. It is known that if $\lim_{n\to\infty} \sum_{k=1}^{n} p_{n,k} = \lambda$ and $\lim_{n\to\infty} \max\{p_{n,k} \mid 1 \leq k \leq n\} = 0$, the sequence (P_{S_n}) converges weakly to the Poisson-distribution mit parameter λ (Poisson limit theorem). Show that in the case of $\lim_{n\to\infty} \sum_{k=1}^{n} p_{n,k}(1 - p_{n,k}) = \infty$ the sequence (S_n) has the central limit property.

Aufgabe H26:

Let μ_n, μ be probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with distribution functions F_n, F. Prove that: If (μ_n) converges weakly to μ and F is continuous, then (F_n) converges uniformly to F on $\mathbb{R}$.

Aufgabe H27:

Suppose that a sequence (X_n) of independent square integrable random variables with variances $\sigma_n^2 > 0$ has the central limit property. Let $s_n := \left(\sum_{k=1}^{n} \sigma_k^2\right)^{1/2}$ and $\alpha_n := \varepsilon n / s_n$. Use H26 to show that

$$P\left[\left|\frac{1}{n}\sum_{i=1}^{n}(X_i - E[X_i])\right| < \varepsilon\right] - \frac{1}{\sqrt{2\pi}} \int_{-\alpha_n}^{\alpha_n} e^{-x^2/2}\, dx$$

converges uniformly in $\varepsilon > 0$ to 0 for $n \to \infty$. Conclude that (X_n) does not satisfy the weak law of large numbers, if the sequence (n/s_n) is bounded.