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Fachbereich Mathematik
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Probability Theory
7. Aufgabenblatt

Gruppenübungen

The following model of a random walk on $\mathbb{Z}$ will be considered in G21, H19 and H21:

Let $\Omega = \{\omega = (x_1, \ldots, x_N \,|\, x_i \in \{-1, 1\}\}$ and let P be the uniform distribution on Ω and $X_i(\omega) = x_i$. Consider the random walk

$$S_n = X_1 + \ldots + X_n \quad (n = 0, 1, \ldots, N)$$

of a particle on $\mathbb{Z}$ with starting point 0. For $0 < a \in \mathbb{N}$ let

$$T_a = \min\{n > 0 \,|\, S_n = a\}$$

be the first time that the particle hits level a.

Aufgabe G21 (Reflection principle):
Prove that

$$P[S_n = a - c, T_a \leq n] = P[S_n = a + c]$$

for every $c \in \mathbb{N}$.

Aufgabe G22 (Arcsin-law):
Let S_n $(n = 1, 2 \ldots, 2N)$ be the random walk from above,

$$T_0(\omega) = \min\{n > 0 \,|\, S_n(\omega) = 0\}$$

be the first time that the particle returns to 0 and

$$L(\omega) = \max\{0 \leq n \neq 2N \,|\, S_n(\omega) = 0\}$$

be the last time, the particle visits 0. For the distribution of L we then have that

$$P[L = 2n] = P[S_{2n} = 0] \cdot P[S_{2N-2n} = 0] = 2^{-2N} \binom{2n}{n} \binom{2N-2n}{N-n}$$

(see H20).
Show that this implies that the distribution μ_N of $\frac{L}{2N}$ for $N \uparrow \infty$ converges weakly to the distribution with density

$$f(x) = \frac{1}{\pi\sqrt{x(1-x)}} \quad (0 < x < 1)$$

and distribution function

$$F(x) = \frac{2}{\pi} \arcsin \sqrt{x}.$$

Hausübungen

Aufgabe H19:

Prove the following two identities for the distribution of T_a:

$$P[T_a \le n] = P[S_n \notin [-a, a-1]],$$
$$P[T_a = n] = \frac{1}{2}(P[S_{n-1} = a-1] - P[S_{n-1} = a+1]) = \frac{a}{n} P[S_n = a]$$

Aufgabe H20:

With the notation of G22 show that

$$P[T_0 > 2n] = P[S_{2n} = 0]$$

and

$$P[L = 2n] = P[S_{2n} = 0] \cdot P[S_{2N-2n} = 0] = 2^{-2N} \binom{2n}{n} \binom{2N-2n}{N-n}$$

("discrete arcsin-distribution").

Aufgabe H21:

Given the random walk S_n, $n \in \mathbb{N}$, on $\mathbb{Z}$. Prove that

$$\limsup_{n \uparrow \infty} S_n = +\infty, \quad \liminf_{n \uparrow \infty} S_n = -\infty \quad \text{P-a.s.}$$