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Fachbereich Mathematik
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Probability Theory
6. Aufgabenblatt

Gruppenübungen

Aufgabe G18:

Determine the Dynkin-system $\mathcal{D}(\mathcal{E})$ generated by $\mathcal{E}$ in the case where $\mathcal{E}$ contains two subsets A and B of Ω. Show that $\mathcal{D}(\mathcal{E})$ and the σ-algebra $\sigma(\mathcal{E})$ generated by $\mathcal{E}$ coincide if and only if $A \cap B$ or $A \cap B^c$ or $A^c \cap B$ or $A^c \cap B^c$ are empty.

Aufgabe G19:

Let X be a real random variable and let f be a real Borel measurable function on $\mathbb{R}$. Show that X and $f \circ X$ are independent if and only if $f \circ X$ are constant a.s.

Aufgabe G20:

Let (A_n) be a sequence of independent events on $(\Omega, \mathcal{A}, P)$ with $P(A_n) < 1$ for all $n \in \mathbb{N}$ and $P(\bigcup_{n=1}^{\infty} A_n) = 1$. Show that

$$\sum_{n=1}^{\infty} P(A_n) = \infty.$$

Hausübungen

Aufgabe H16:

Let P be the uniform distribution on $\Omega = \{0,1\}^N$. For $j = 1, \ldots, N$ define $A_j = \{(\omega_1, \ldots, \omega_N) \in \Omega \,|\, \omega_j = 1\}$ and $A_{N+1} = \{(\omega_1, \ldots, \omega_N) \in \Omega \,|\, \omega_1 + \ldots + \omega_N \text{ even}\}$. Show that $A_1, \ldots, A_{N+1}$ are dependent but any N events out of $A_1, \ldots, A_{N+1}$ are independent.

Aufgabe H17:

Let (A_n) be a sequence of independent events and $p_n := P(A_n)$. Which assumptions for p_n imply $\lim_{n\to\infty} 1_{A_n} = 0$

(i) in probability,

(ii) P-a.s.?

Hint: Use the lemma of Borel-Cantelli.

Aufgabe H18:

As in H5 consider the model for infinitely many fair coin tosses and

$$\ell_n((x_n)_{n\in\mathbb{N}}) := \min\{k \geq 1 \,|\, x_n = \ldots = x_{n+k-1} = 1, x_k = 0\}$$

For $r \geq 0$ define $E_n(r) := \{\ell_n \geq r\}$.

(i) With the lemma of Borel-Cantelli show for any increasing sequence of non-negative real numbers $r_1, r_2, \ldots$ with $\sum_{n=1}^{\infty} \frac{2^{-r_n}}{r_n} = \infty$ that

$$P\left[\limsup_{n\to\infty} E_n(r_n)\right] = 1.$$

(ii) Use (i) to conclude that $P[\limsup_{n\to\infty} E_n(\log_2 n)] = 1$ and therefore with H5 that

$$P\left[\limsup_{n\to\infty} \frac{\ell_n}{\log_2 n} = 1\right] = 1.$$

(Hint for (i): Define the sequence (n_k) inductively with $n_1 = 1$ and $n_{k+1} = n_k + r_{n_k}$. Then the events $E_{n_k}(r_{n_k}), k \geq 1$, are independent.)