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Probability Theory
5. Aufgabenblatt

Gruppenübungen

Aufgabe G15:

Let $(X_i)_{i\in I}$ and $(Y_i)_{i\in I}$ be random variables on a probability space $(\Omega, \mathcal{A}, P)$ which are uniformly integrable and $\alpha, \beta \in \mathbb{R}$. Show that $(\alpha X_i + \beta Y_i)_{i\in I}$ is uniformly integrable, too.

Aufgabe G16:

Consider a probability measure P on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and define the cumulative distribution function (cdf) $F : \mathbb{R} \to [0, 1]$ by $F(x) = P(] - \infty, x[)$. Show that in this exercise only(!)

(i) F is non-decreasing,

(ii) F is left continuous,

(iii) $\lim_{x\to\infty} F(x) = 1$, $\lim_{x\to-\infty} F(x) = 0$,

(iv) for all $x \in \mathbb{R}$, $\lim_{y\downarrow x} F(y) - \lim_{y\uparrow x} F(y) = P(\{x\}) = 0$.

Aufgabe G17:

(a) Recall that if $X_n \to X$ in probability then $P_{X_n} \to P_X$ weakly. Show that if X is P-a.s. constant, then the converse is also true.

(b) Give an example for which $\lim_{n\to\infty} P_{X_n} = P_X$ weakly, but $(X_n)_n$ is not converging in probability.

Hausübungen

Aufgabe H13:

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables on $(\Omega, \mathcal{A}, P)$ with $X_n \to X$ in $\mathcal{L}^1$. Show that $(X_n)_{n\in\mathbb{N}}$ is uniformly integrable.

Aufgabe H14:

Let $\Omega =]0,1]$ and $P = \lambda|_{\mathcal{A}}$ be the Lebesgue measure on $\mathcal{A} = \mathcal{B}(]0,1])$. Define

$$A_{2^i+k} :=]\frac{k}{2^i}, \frac{k+1}{2^i}], \quad 0 \le k < 2^i, \quad i \in \mathbb{N}_0,$$

$$Y_n := 1_{A_n}, \quad \widetilde{Y}_n := n^{\frac{1}{p}} \cdot 1_{]0,\frac{1}{n}[}, \quad n \in \mathbb{N}.$$

Prove that:

(i) $Y_n \to 0$ in $\mathcal{L}^p$ for $p > 0$ ($\Rightarrow Y_n \to 0$ in probability).

(ii) $Y_n \not\to 0$ P-a.s.

(iii) $\widetilde{Y}_n \to 0$ P-a.s. ($\Rightarrow \widetilde{Y}_n \to 0$ in probability).

(iv) $\widetilde{Y}_n \not\to 0$ in $\mathcal{L}^p$ for $p > 0$.

Aufgabe H15:

Let μ_n, μ be probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and assume that the sequence (μ_n) converges vaguely to μ, i.e., $\lim_{n\to\infty} \int f\, d\mu_n = \int f\, d\mu$ for all continuous real functions f on $\mathbb{R}$ with compact support. Show that (μ_n) converges to μ weakly.