TU Darmstadt
Fachbereich Mathematik
Wilhelm Stannat

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Probability Theory
4. Aufgabenblatt

Gruppenübungen

Aufgabe G11:

Let X be a rondom variable on $(\Omega, \mathcal{A}, P)$ with values in $\mathbb{N}$. Prove that

$$E(X) = \sum_{n=1}^{\infty} P[X \geq n].$$

Aufgabe G12:

Let $(\Omega, \mathcal{A}, P)$ be a probability space and X be a random variable. Prove that

(a) $E(|X|) = 0 \Rightarrow X = 0$ P-a.s.

(b) $E(|X|) < \infty \Rightarrow |X| < \infty$ P-a.s.

Aufgabe G13:

Let $(\Omega, \mathcal{A}, P)$ be a probability space. Prove that for $f, g \in \mathcal{L}^1(\Omega, \mathcal{A}, P)$:

(a) $f < g$ P-a.s. $\Longleftrightarrow \int_A f\,dP < \int_A g\,dP \quad \forall A \in \mathcal{A}$ with $P(A) > 0$.

(b) $f \leq g$ P-a.s. $\Longleftrightarrow \int_A f\,dP \leq \int_A g\,dP \quad \forall A \in \mathcal{A}$.

Aufgabe G14:

Let $X_1, X_2, \ldots$ be a sequence of random variables with $E[X_i] = m$ and $Var(X_i) = \sigma^2$. Suppose there exists a function r on $\mathbb{N}$ with $|Cov(X_i, X_j)| \leq r(|i-j|)$. Find properties for r which still imply the Weak Law of Large Numbers

$$\lim_{n\to\infty} P\left[\left|\frac{X_1 + X_2 + \ldots + X_n}{n} - m\right| > \epsilon\right] = 0 \qquad \forall \epsilon > 0.$$

Hausübungen

Aufgabe H10:

Prove the following generalization of Lebesgue's theorem. Let $(X_n), (Y_n)$ be positive random variables on a probability space $(\Omega, \mathcal{A}, P)$ with $X_n \leq Y_n$ for all $n \in \mathbb{N}$. Show that if $\lim_{n\to\infty} X_n = X$, $\lim_{n\to\infty} Y_n = Y$ and $\lim_{n\to\infty} E(Y_n) = E(Y) < \infty$ it follows that $\lim_{n\to\infty} E(X_n) = E(X)$.

Aufgabe H11:

For $x \in [0,1]$ let $x = \sum_{n=1}^{\infty} \frac{d_n(x)}{2^n}$ denote its binary expansion. A number x is called **normal** if the sequence of the relativ frequencies of 1 in its binary expansion converges to $\frac{1}{2}$, i.e.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} d_i(x) = \frac{1}{2}.$$

Use the Strong Law of Large Numbers to prove that the set N of normal numbers in $[0,1]$ has Lebesgue measure 1.

Aufgabe H12:

Prove the following Weak Law of Large Numbers:
If $0 < a_1 \leq a_2 \leq \ldots \nearrow \infty$ and $(X_n)_{n\in\mathbb{N}}$ are pairwise uncorrelated with $\sum\limits_{k=1}^{\infty} \frac{Var(X_k)}{a_k^2} < \infty$, then

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{k=1}^{n} (X_k - E(X_k)) = 0 \quad \text{in probability.}$$