TU Darmstadt
Fachbereich Mathematik
Wilhelm Stannat

WS 2007/08
24.10.07

Probability Theory
3. Aufgabenblatt

Gruppenübungen

Aufgabe G8:

Let Ω be the set of permutations of $\{1,\dots,n\}$ and P be the uniform distribution on Ω. For a permutation ω let $X(\omega)$ denote the number of fixed points. Calculate the expectation $E[X]$ and the variance $Var(X) = E[X^2] - E[X]^2$.

Aufgabe G9:

Let $(\Omega_i, \mathcal{A}_i, P_i)$, $i = 1, 2$, be probability spaces and P_2 the image of P_1 with respect to the mapping $T : \Omega_1 \to \Omega_2$. Show for every random variable $X_2 \geq 0$ on $(\Omega_2, \mathcal{A}_2)$

$$E_2[X_2] = E_1[X_2 \circ T].$$

Aufgabe G10:

Let X be a random variable with the property

$$P[X \leq x] \in \{0, 1\}$$

for all $x \in \mathbb{R}$. Show that there exists a constant $a \in \mathbb{R}$ with:

(a) $X = a$ P-a.s.

(b) $X(P) = \delta_a$.

Hausübungen

Aufgabe H7:

Let $(\Omega, \mathcal{A})$ be a measurable space and $D \subset \mathbb{R}$ be a dense subset. Prove that for a function $X : \Omega \to \overline{\mathbb{R}}$ the following statements are equivalent:

(i) X is $\mathcal{A}$-measurable,

(ii) $\{X \geq \alpha\} \in \mathcal{A} \quad \forall \alpha \in D$,

(iii) $\{X > \alpha\} \in \mathcal{A} \quad \forall \alpha \in D$,

(iv) $\{X \leq \alpha\} \in \mathcal{A} \quad \forall \alpha \in D$,

(v) $\{X < \alpha\} \in \mathcal{A} \quad \forall \alpha \in D$.

Aufgabe H8:

Let $(\Omega, \mathcal{A})$ be a measurable space.

(a) Let $X : \Omega \to \overline{\mathbb{R}}$ be a function. Show that X is $\mathcal{A}$-measurable if and only if $X^{-1}(\{+\infty\}) \in \mathcal{A}$ and $X^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{B}(\mathbb{R})$.

(b) Give an example for a function $X : \mathbb{R} \to \overline{\mathbb{R}}$ for which $|X|$ is Borel measurable but X is not Borel measurable.

(c) Show that for $\mathcal{A}$-measurable functions $X_n : \Omega \to \overline{\mathbb{R}}$, $n \in \mathbb{N}$, the sets

$$A := \{\omega \in \Omega \mid \lim_{n\to\infty} X_n(\omega) \text{ exists in } \mathbb{R}\}$$

and

$$\overline{A} := \{\omega \in \Omega \mid \lim_{n\to\infty} X_n(\omega) \text{ exists in } \overline{\mathbb{R}}\}$$

are $\mathcal{A}$-measurable.

Aufgabe H9:

Let $(\mathbb{R}, \mathcal{A}, \mu)$ be the probability space with $\mathcal{A} := \{A \subset \mathbb{R} \mid A \text{ or } A^c \text{ countable}\}$ and

$$\mu(A) := \begin{cases} 1 & \text{if } A^c \text{ countable}, \\ 0 & \text{if } A \text{ countable}. \end{cases}$$

Define $\Omega' := \{0, 1\}$, $\mathcal{A}' := \mathcal{P}(\Omega')$ and the mapping $T : \mathbb{R} \to \Omega'$ by

$$T(x) := \begin{cases} 1 & \text{if } x \in \mathbb{R}\backslash\mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{Q}. \end{cases}$$

Show that T is $\mathcal{A}/\mathcal{A}'$-measurable and determine $T(\mu)$.