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Probability Theory
2. Aufgabenblatt

Gruppenübungen

Aufgabe G5:

Let $\mathcal{A}_i$ be σ-algebras of subsets of Ω_i, $i = 1, 2$, and $T : \Omega_1 \to \Omega_2$ be a mapping. Prove:

(i) $\{T^{-1}(B) \mid B \in \mathcal{A}_2\}$ is the smallest σ-algebra $\mathcal{A}$ of subsets of Ω_1 for which T is $\mathcal{A}/\mathcal{A}_2$-measurable.

(ii) $\{B \subset \Omega_2 \mid T^{-1}(B) \in \mathcal{A}_1\}$ is the largest σ-algebra $\mathcal{A}'$ of subsets of Ω_2 for which T is $\mathcal{A}_1/\mathcal{A}'$-measurable.

Aufgabe G6:

Show that every continuous mapping $f : \mathbb{R}^d \to \mathbb{R}^{d'}$ is $\mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathbb{R}^{d'})$-measurable.

Aufgabe G7 (Factorization lemma):

Let $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$ be measurable spaces, $T : \Omega \to \Omega'$ be a $\mathcal{A}/\mathcal{A}'$-measurable mapping and $\sigma(T)$ the σ-algebra of subsets of Ω generated by T. Show that every $\sigma(T)$-measurable random variable $X : \Omega \to \mathbb{R}$ can be written as

$$X = f(T)$$

with $f : \Omega' \to \mathbb{R}$ measurable.

Hausübungen

Aufgabe H5:

Consider the model for infinitely many fair coin tosses from example 3.6. For $n \in \mathbb{N}$ let

$$\ell_n((x_n)_{n\in\mathbb{N}}) := \max\{k \geq 1 | x_n = \ldots = x_{n+k-1} = 1\}$$

be the number of consecutive ones starting from the n-th coin toss ("run"). Let $\max \emptyset =: 0$. For a given sequence $r_1, r_2, \ldots \in \mathbb{N}_0$ consider the events $E_n = \{\ell_n \geq r_n\}$.

(i) Show with the lemma of Borel-Cantelli that

$$P[\ell_n \geq r_n \text{ infinitely often }] = 0$$

if $\sum_{n=1}^{\infty} 2^{-r_n} < \infty$.

(ii) For the particular sequence $r_n = (1+\epsilon)\log_2 n$, $\epsilon > 0$, (i) implies that $P[\ell_n \geq (1+\epsilon)\log_2 n \text{ infinitely often }] = 0$. With this show that

$$P[\limsup_{n\to\infty} \frac{\ell_n}{\log_2 n} > 1] = 0.$$

Aufgabe H6:

Let P be the Lebesgue-measure on $[0,1]$. For $a \in [0,1]$ let $T_a : [0,1] \to \mathbb{R}$ be defined by

$$T_a(x) = \begin{cases} a + x & \text{if } a + x \leq 1, \\ a + x - 1 & \text{if } a + x > 1. \end{cases}$$

(a) Show that T_a is a measurable bijection of $[0,1]$ and $T_a(P) = P$.

(b) $a, b \in [0,1]$ should be called equivalent if $a - b$ is a rational number. Show that this is indeed an equivalence relation.

(c) Let M be a subset of $[0,1]$ which contains exacly one element from every equivalence class. Show that the sets $T_a(M)$, $a \in [0,1] \cap \mathbb{Q}$, form a partition of [0,1].

(d) Show that M is not a Borel subset of [0,1].