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Probability Theory
1. Aufgabenblatt

Gruppenübungen

Aufgabe G1:

Let $\mathcal{A}$ be a collection of subsets of Ω. Prove that $\mathcal{A}$ is an algebra if and only if $\Omega \in \mathcal{A}$, $A^c \in \mathcal{A}$ if $A \in \mathcal{A}$, and $A \cup B \in \mathcal{A}$ if $A, B \in \mathcal{A}$.

Aufgabe G2:

Which of the following algebras of subsets of Ω is a σ-algebra?

(a) $\mathcal{A}_1 = \{A \subset \Omega \mid A \text{ or } A^c \text{ is finite}\}$

(b) $\mathcal{A}_2 = \{A \subset \Omega \mid A \text{ or } A^c \text{ is countable}\}$

Aufgabe G3:

Let $\mathcal{A}_i$ be σ-algebras of subsets of Ω_i, $i = 1, 2$, and $T : \Omega_1 \to \Omega_2$ be a mapping. Prove that $\{T^{-1}(B) | B \in \mathcal{A}_2\}$ and $\{B \subset \Omega_2 | T^{-1}(B) \in \mathcal{A}_1\}$ are σ-algebras.

Aufgabe G4:

Let Ω be an infinite set.

(i) Let $\mathcal{A}_1$ be the algebra defined in G2 (a) and let Ω be countable. Show that the set function

$$\mu(A) = \begin{cases} 0 & \text{if } A \ \text{ finite} \\ +\infty & \text{if } A^c \text{ finite} \end{cases}$$

is finitely additive, but not σ-additive.

(ii) Let $\mathcal{A}_2$ be the algebra defined in G2 (b) and let Ω be uncountable. Prove that the set function

$$\mu(A) = \begin{cases} 0 & \text{if } A \ \text{ countable} \\ 1 & \text{if } A^c \text{ countable} \end{cases}$$

is a measure.

Hausübungen

Aufgabe H1:

Let $\mathcal{A}$ be the Borel σ-algebra on the closed interval $[0,1]$. Prove that

$$\mathcal{A} = \sigma\left(\{[a,b] : 0 \le a < b \le 1\}\right) .$$

Aufgabe H2:

Let $\mathcal{A}$ be a σ-algebra of subsets of Ω, $P : \mathcal{A} \to \mathbb{R}_+$ with $P(\Omega) = 1$. Prove that the following statements are equivalent:

(a) P is a probability measure.

(b) P is finitely additive and $\bigcap_{n\in\mathbb{N}} A_n \neq \emptyset$ for any sequence (A_n) of subsets of $\mathcal{A}$ with $A_n \supset A_{n+1}$ for all $n \in \mathbb{N}$ and $\inf_{n\in\mathbb{N}} P(A_n) > 0$.

Aufgabe H3:

Let $\mathcal{P}$ be the set of all probability measures on a countable set Ω.

(i) Show that $\mathcal{P}$ is convex.

(ii) Characterize the extremal points of $\mathcal{P}$.

(iii) Prove that every $P \in \mathcal{P}$ can be written as a convex combination

$$P = \sum_i \alpha_i P_i\,, \qquad \alpha_i \ge 0\,, \sum_i \alpha_i = 1$$

with P_i extremal.

Aufgabe H4:

Recall that the hypergeometric distribution with parameters N, K, $n \in \mathbb{N}$, $K, n \le N$, is given by

$$\frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}\,, \qquad k = 0, \dots, n\,.$$

Prove that for $N \to \infty$ and $K \to \infty$ with $p = \frac{K}{N}$ constant, the hypergeometric distribution converges towards the binomial distribution with parameter n, p.