**Example 5.** Let $I = \mathbb{R}_+$, $\Omega_i = \mathbb{R}$, and $\mathfrak{A}_i = \mathfrak{B}$. For the corresponding product space $(\Omega, \mathfrak{A})$ we have $\Omega = \mathbb{R}^{\mathbb{R}_+}$ and

$$|\mathfrak{A}| = |\mathbb{R}| < |\Omega|.$$

Proof: Clearly $|\mathbb{R}| \leq |\mathfrak{A}|$ and $|\mathbb{R}| < |\Omega|$. On the other hand, Theorem 2 shows that $\mathfrak{A} = \sigma(\mathfrak{E})$ for some set $\mathfrak{E}$ with $|\mathfrak{E}| = |\mathbb{R}|$. Hence $|\mathfrak{A}| \leq |\mathbb{R}|$ by Theorem 4.

The space $\mathbb{R}^{\mathbb{R}_+}$ already appeared in the introductory Example I.3. The product $\sigma$-algebra $\mathfrak{A} = \bigotimes_{i \in \mathbb{R}_+} \mathfrak{B}$ is a proper choice on this space. On the subspace $C(\mathbb{R}_+) \subset \mathbb{R}^{\mathbb{R}_+}$ we can take the trace-$\sigma$-algebra. It is important to note, however, that

$$C(\mathbb{R}_+) \notin \mathfrak{A},$$

see Übung 3.2. It turns out that the Borel $\sigma$-algebra $\mathfrak{B}(C(\mathbb{R}_+))$ that is generated by the topology of uniform convergence on compact intervals coincides with the trace-$\sigma$-algebra of $\mathfrak{A}$ in $C(\mathbb{R}_+)$, see Bauer (1996, Theorem 38.6).

# 4 Construction of (Probability) Measures

Given: $\Omega \neq \emptyset$ and $\emptyset \neq \mathfrak{A} \subset \mathfrak{P}(\Omega)$.

**Definition 1.** $\mu : \mathfrak{A} \to \mathbb{R}_+ \cup \{\infty\}$ is called

(i) *additive* if:

$$A, B \in \mathfrak{A} \wedge A \cap B = \emptyset \wedge A \cup B \in \mathfrak{A} \quad \Rightarrow \quad \mu(A \cup B) = \mu(A) + \mu(B),$$

(ii) *$\sigma$-additive* if

$$A_1, A_2, \ldots \in \mathfrak{A} \text{ pairwise disjoint } \wedge \bigcup_{i=1}^{\infty} A_i \in \mathfrak{A} \quad \Rightarrow \quad \mu\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} \mu(A_i),$$

(iii) *content (on $\mathfrak{A}$)* if

$$\mathfrak{A} \text{ algebra} \quad \wedge \quad \mu \text{ additive} \quad \wedge \quad \mu(\emptyset) = 0,$$

(iv) *pre-measure (on $\mathfrak{A}$)* if

$$\mathfrak{A} \text{ semi-algebra} \quad \wedge \quad \mu\ \sigma\text{-additive} \quad \wedge \quad \mu(\emptyset) = 0,$$

(v) *measure (on $\mathfrak{A}$)* if

$$\mathfrak{A}\ \sigma\text{-algebra} \quad \wedge \quad \mu \text{ pre-measure},$$

(vi) *probability measure (on $\mathfrak{A}$)* if

$$\mu \text{ measure} \quad \wedge \quad \mu(\Omega) = 1.$$

**Definition 2.** $(\Omega, \mathfrak{A}, \mu)$ is called a

(i) *measure space*, if $\mu$ is a measure on the $\sigma$-algebra $\mathfrak{A}$ in $\Omega$,

(ii) *probability space*, if $\mu$ is a probability measure on the $\sigma$-algebra $\mathfrak{A}$ in $\Omega$.

**Example 1.**

(i) *Lebesgue pre-measure* $\lambda_1$ on the class $\mathfrak{I}_1$ of intervals from Example 1.1.(i): $\lambda_1(A)$ is the length of $A \in \mathfrak{I}_1$, i.e.,

$$\lambda_1(]a, b]) = b - a$$

if $a, b \in \mathbb{R}$ with $a \leq b$ and $\lambda_1(A) = \infty$ if $A \in \mathfrak{I}_1$ is unbounded. See Billingsley (1979, p. 22), Elstrodt (1996, §II.2), or Analysis IV.

Analogously for cartesian products of such intervals. Hereby we get the semi-algebra $\mathfrak{I}_k$ of rectangles in $\mathbb{R}^k$. The *Lebesgue pre-measure* $\lambda_k$ on $\mathfrak{I}_k$ yields the volume $\lambda_k(A)$ of $A \in \mathfrak{I}_k$, i.e., the product of the side-lengths of $A$. See Elstrodt (1996, §III.2) or Analysis IV.

(ii) for any semi-algebra $\mathfrak{A}$ in $\Omega$ and $\omega \in \Omega$

$$\varepsilon_\omega(A) = 1_A(\omega), \qquad A \in \mathfrak{A},$$

defines a pre-measure. If $\mathfrak{A}$ is a $\sigma$-algebra, then $\varepsilon_\omega$ is called the *Dirac measure* at the point $\omega$.

More generally: take sequences $(\omega_n)_{n\in\mathbb{N}}$ in $\Omega$ and $(\alpha_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+$ such that $\sum_{n=1}^\infty \alpha_n = 1$. Then

$$\mu(A) = \sum_{n=1}^{\infty} \alpha_n \cdot 1_A(\omega_n), \qquad A \in \mathfrak{A},$$

defines a *discrete probability measure* on any $\sigma$-algebra $\mathfrak{A}$ in $\Omega$. Note that $\mu = \sum_{n=1}^\infty \alpha_n \cdot \varepsilon_{\omega_n}$.

(iii) *Counting measure* on a $\sigma$-algebra $\mathfrak{A}$

$$\mu(A) = |A|, \qquad A \in \mathfrak{A}.$$

*Uniform distribution* in the case $|\Omega| < \infty$ and $\mathfrak{A} = \mathfrak{P}(\Omega)$

$$\mu(A) = \frac{|A|}{|\Omega|}, \qquad A \subset \Omega.$$

(iv) On the algebra $\mathfrak{A} = \{A \subset \Omega : A \text{ finite or } A^c \text{ finite}\}$ let

$$\mu(A) = \begin{cases} 0 & \text{if } |A| < \infty \\ \infty & \text{if } |A| = \infty. \end{cases}$$

Then $\mu$ is a content but not a pre-measure in general.

(v) For the semi-algebra of measurable rectangles in Example 3.1 and $A_i \subset \{0,1\}$

$$\mu(A_1 \times \cdots \times A_n \times \Omega_{n+1} \times \cdots) = \frac{|A_1 \times \ldots \times A_n|}{|\{0,1\}^n|}$$

is well defined and yields a pre-measure $\mu$ with $\mu\big(\{0,1\}^{\mathbb{N}}\big) = 1$.

**Remark 1.** For every content $\mu$ on $\mathfrak{A}$ and $A, B \in \mathfrak{A}$

(i) $A \subset B \Rightarrow \mu(A) \leq \mu(B)$ (*monotonicity*),

(ii) $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$,

(iii) $A \subset B \wedge \mu(A) < \infty \Rightarrow \mu(B \setminus A) = \mu(B) - \mu(A)$,

(iv) $\mu(A) < \infty \wedge \mu(B) < \infty \Rightarrow |\mu(A) - \mu(B)| \leq \mu(A \triangle B)$,

(v) $\mu(A \cup B) \leq \mu(A) + \mu(B)$ (*subadditivity*).

To proof these facts use, for instance, $A \cup B = A \cup (B \cap A^c)$.

**Theorem 1.** Consider the following properties for a content $\mu$ on $\mathfrak{A}$:

(i) $\mu$ pre-measure,

(ii) $A_1, A_2, \ldots \in \mathfrak{A} \wedge \bigcup_{i=1}^{\infty} A_i \in \mathfrak{A} \Rightarrow \mu\big(\bigcup_{i=1}^{\infty} A_i\big) \leq \sum_{i=1}^{\infty} \mu(A_i)$ (*σ-subadditivity*),

(iii) $A_1, A_2, \ldots \in \mathfrak{A} \wedge A_n \uparrow A \in \mathfrak{A} \Rightarrow \lim_{n\to\infty} \mu(A_n) = \mu(A)$ (*σ-continuity from below*),

(iv) $A_1, A_2, \ldots \in \mathfrak{A} \wedge A_n \downarrow A \in \mathfrak{A} \wedge \mu(A_1) < \infty \Rightarrow \lim_{n\to\infty} \mu(A_n) = \mu(A)$ (*σ-continuity from above*),

(v) $A_1, A_2, \ldots \in \mathfrak{A} \wedge A_n \downarrow \emptyset \wedge \mu(A_1) < \infty \Rightarrow \lim_{n\to\infty} \mu(A_n) = 0$ (*σ-continuity at ∅*).

Then

$$\text{(i)} \Leftrightarrow \text{(ii)} \Leftrightarrow \text{(iii)} \Rightarrow \text{(iv)} \Leftrightarrow \text{(v)}.$$

If $\mu(\Omega) < \infty$, then (iii) $\Leftrightarrow$ (iv).

*Proof.* '(i) $\Rightarrow$ (ii)': Put $B_m = \bigcup_{i=1}^{m} A_i$ and $B_0 = \emptyset$. Then

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{m=1}^{\infty} (B_m \setminus B_{m-1})$$

with pairwise disjoint sets $B_m \setminus B_{m-1} \in \mathfrak{A}$. Clearly $B_m \setminus B_{m-1} \subset A_m$. Hence, by Remark 1.(i),

$$\mu\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{m=1}^{\infty} \mu(B_m \setminus B_{m-1}) \leq \sum_{m=1}^{\infty} \mu(A_m).$$

'(ii) $\Rightarrow$ (i)': Let $A_1, A_2, \ldots \in \mathfrak{A}$ be pairwise disjoint with $\bigcup_{i=1}^\infty A_i \in \mathfrak{A}$. Then

$$\mu\Big(\bigcup_{i=1}^{\infty} A_i\Big) \geq \mu\Big(\bigcup_{i=1}^{n} A_i\Big) = \sum_{i=1}^{n} \mu(A_i),$$

and therefore

$$\sum_{i=1}^{\infty} \mu(A_i) \leq \mu\Big(\bigcup_{i=1}^{\infty} A_i\Big).$$

The reverse estimate holds by assumption.

'(i) $\Rightarrow$ (iii)': Put $A_0 = \emptyset$ and $B_m = A_m \setminus A_{m-1}$. Then

$$\mu\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{m=1}^{\infty} \mu(B_m) = \lim_{n\to\infty} \sum_{m=1}^{n} \mu(B_m) = \lim_{n\to\infty} \mu\Big(\bigcup_{m=1}^{n} B_m\Big) = \lim_{n\to\infty} \mu(A_n).$$

'(iii) $\Rightarrow$ (i)': Let $A_1, A_2, \ldots \in \mathfrak{A}$ be pairwise disjoint with $\bigcup_{i=1}^\infty A_i \in \mathfrak{A}$, and put $B_m = \bigcup_{i=1}^m A_i$. Then $B_m \uparrow \bigcup_{i=1}^\infty A_i$ and

$$\mu\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \lim_{m\to\infty} \mu(B_m) = \sum_{i=1}^{\infty} \mu(A_i).$$

'(iv) $\Rightarrow$ (v)' trivially holds.

'(v) $\Rightarrow$ (iv)': Use $B_n = A_n \setminus A \downarrow \emptyset$.

'(i)' $\Rightarrow$ (v)': Note that $\mu(A_1) = \sum_{i=1}^\infty \mu(A_i \setminus A_{i+1})$. Hence

$$0 = \lim_{k\to\infty} \sum_{i=k}^{\infty} \mu(A_i \setminus A_{i+1}) = \lim_{k\to\infty} \mu(A_k).$$

'(iv) $\wedge$ $\mu(\Omega) < \infty \Rightarrow$ (iii)': Clearly $A_n \uparrow A$ implies $A_n^c \downarrow A^c$. Thus

$$\mu(A) = \mu(\Omega) - \mu(A^c) = \lim_{n\to\infty} (\mu(\Omega) - \mu(A_n^c)) = \lim_{n\to\infty} \mu(A_n).$$

$\square$

**Theorem 2** (Extension: semi-algebra $\rightsquigarrow$ algebra)**.** For every semi-algebra $\mathfrak{A}$ and every additive mapping $\mu : \mathfrak{A} \to \mathbb{R}_+ \cup \{\infty\}$ with $\mu(\emptyset) = 0$

$$\underset{1}{\exists}\, \widehat{\mu} \text{ content on } \alpha(\mathfrak{A}) : \quad \widehat{\mu}|_{\mathfrak{A}} = \mu.$$

Moreover, if $\mu$ is $\sigma$-additive then $\widehat{\mu}$ is $\sigma$-additive, too.

*Proof.* We have $\alpha(\mathfrak{A}) = \mathfrak{A}^+$, see Lemma 1.1. Necessarily

$$\widehat{\mu}\Big(\bigcup_{i=1}^{n} A_i\Big) = \sum_{i=1}^{n} \mu(A_i) \tag{1}$$

for $A_1, \ldots, A_n \in \mathfrak{A}$ pairwise disjoint. Use (1) to obtain a well-defined extension of $\mu$ onto $\alpha(\mathfrak{A})$. It remains to verify that $\widehat{\mu}$ is additive or even $\sigma$-additive. $\square$

**Example 2.** For the semi-algebra $\mathfrak{A}$ in Example 1.(v) $\alpha(\mathfrak{A})$ is the algebra of cylinder sets, and

$$\widehat{\mu}(A \times \Omega_{n+1} \times \cdots) = \frac{|A|}{|\{0,1\}^n|}, \qquad A \subset \{0,1\}^n .$$

**Theorem 3** (Extension: algebra $\rightsquigarrow$ $\sigma$-algebra, Carathéodory)**.** For every pre-measure $\mu$ on an algebra $\mathfrak{A}$

$$\exists\, \mu^* \text{ measure on } \sigma(\mathfrak{A}): \quad \mu^*|_{\mathfrak{A}} = \mu.$$

*Proof.* Define $\overline{\mu} : \mathfrak{P}(\Omega) \to \mathbb{R}_+ \cup \{\infty\}$ by

$$\overline{\mu}(A) = \inf\Big\{\sum_{i=1}^{\infty} \mu(A_i) : A_i \in \mathfrak{A},\ A \subset \bigcup_{i=1}^{\infty} A_i\Big\}.$$

Then $\overline{\mu}$ is an *outer measure*, i.e., $\overline{\mu}(\emptyset) = 0$ and $\overline{\mu}$ is monotone and $\sigma$-subadditive, see Billingsley (1979, Exmp. 11.1) and compare Analysis IV. Actually it suffices to have $\mu \geq 0$ and $\emptyset \in \mathfrak{A}$ with $\mu(\emptyset) = 0$.

We claim that

(i) $\overline{\mu}|_{\mathfrak{A}} = \mu$,

(ii) $\forall\, A \in \mathfrak{A}\ \forall\, B \in \mathfrak{P}(\Omega): \quad \overline{\mu}(B) = \overline{\mu}(B \cap A) + \overline{\mu}(B \cap A^c)$.

Ad (i): For $A \in \mathfrak{A}$

$$\overline{\mu}(A) \leq \mu(A) + \sum_{i=2}^{\infty} \mu(\emptyset) = \mu(A),$$

and for $A_i \in \mathfrak{A}$ with $A \subset \bigcup_{i=1}^{\infty} A_i$

$$\mu(A) = \mu\Big(\bigcup_{i=1}^{\infty}(A_i \cap A)\Big) \leq \sum_{i=1}^{\infty} \mu(A_i \cap A) \leq \sum_{i=1}^{\infty} \mu(A_i)$$

follows from Theorem 1.(ii).

Ad (ii): '$\leq$' holds due to sub-additivity of $\overline{\mu}$, and '$\geq$' is easily verified.

Consider the class

$$\overline{\mathfrak{A}} = \overline{\mathfrak{A}}_{\overline{\mu}} = \{A \in \mathfrak{P}(\Omega) : \forall\, B \in \mathfrak{P}(\Omega) : \overline{\mu}(B) = \overline{\mu}(B \cap A) + \overline{\mu}(B \cap A^c)\}$$

of so-called $\overline{\mu}$-measurable sets.

We claim that

(iii) $\forall\, A_1, A_2 \in \overline{\mathfrak{A}}\ \forall\, B \in \mathfrak{P}(\Omega): \quad \overline{\mu}(B) = \overline{\mu}(B \cap (A_1 \cap A_2)) + \overline{\mu}(B \cap (A_1 \cap A_2)^c)$.

(iv) $\overline{\mathfrak{A}}$ algebra,

Ad (iii): We have

$$\begin{aligned}\overline{\mu}(B) &= \overline{\mu}(B \cap A_1) + \overline{\mu}(B \cap A_1^c) \\ &= \overline{\mu}(B \cap A_1 \cap A_2) + \overline{\mu}(B \cap A_1 \cap A_2^c) + \overline{\mu}(B \cap A_1^c)\end{aligned}$$

and

$$\overline{\mu}(B \cap (A_1 \cap A_2)^c) = \overline{\mu}(B \cap A_1^c \cup B \cap A_2^c) = \overline{\mu}(B \cap A_2^c \cap A_1) + \overline{\mu}(B \cap A_1^c).$$

Ad (iv): Cleary $\Omega \in \overline{\mathfrak{A}}$, $A \in \overline{\mathfrak{A}} \Rightarrow A^c \in \overline{\mathfrak{A}}$, and $\overline{\mathfrak{A}}$ is closed w.r.t. intersections by (iii). We claim that

(v) $\forall A_1, A_2 \in \overline{\mathfrak{A}}$ disjoint $\forall B \in \mathfrak{P}(\Omega)$ : $\quad \overline{\mu}(B \cap (A_1 \cup A_2)) = \overline{\mu}(B \cap A_1) + \overline{\mu}(B \cap A_2)$.

In fact, since $A_1 \cap A_2 = \emptyset$,

$$\overline{\mu}(B \cap (A_1 \cup A_2)) = \overline{\mu}(B \cap A_1) + \overline{\mu}(B \cap A_2 \cap A_1^c) = \overline{\mu}(B \cap A_1) + \overline{\mu}(B \cap A_2).$$

We claim that

(vi) $\forall A_1, A_2, \ldots \in \overline{\mathfrak{A}}$ pairwise disjoint

$$\bigcup_{i=1}^{\infty} A_i \in \overline{\mathfrak{A}} \quad \wedge \quad \overline{\mu}\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} \overline{\mu}(A_i).$$

Let $B \in \mathfrak{P}(\Omega)$. By (iv), (v), and monotonicity of $\overline{\mu}$

$$\begin{aligned}\overline{\mu}(B) &= \overline{\mu}\Big(B \cap \bigcup_{i=1}^{n} A_i\Big) + \overline{\mu}\Big(B \cap \Big(\bigcup_{i=1}^{n} A_i\Big)^c\Big) \\ &\geq \sum_{i=1}^{n} \overline{\mu}(B \cap A_i) + \overline{\mu}\Big(B \cap \Big(\bigcup_{i=1}^{\infty} A_i\Big)^c\Big).\end{aligned}$$

Use $\sigma$-subadditivity of $\overline{\mu}$ to get

$$\begin{aligned}\overline{\mu}(B) &\geq \sum_{i=1}^{\infty} \overline{\mu}(B \cap A_i) + \overline{\mu}\Big(B \cap \Big(\bigcup_{i=1}^{\infty} A_i\Big)^c\Big) \\ &\geq \overline{\mu}\Big(B \cap \bigcup_{i=1}^{\infty} A_i\Big) + \overline{\mu}\Big(B \cap \Big(\bigcup_{i=1}^{\infty} A_i\Big)^c\Big) \\ &\geq \overline{\mu}(B).\end{aligned}$$

Hence $\bigcup_{i=1}^{\infty} A_i \in \overline{\mathfrak{A}}$. Take $B = \bigcup_{i=1}^{\infty} A_i$ to obtain $\sigma$-additivity of $\overline{\mu}|_{\overline{\mathfrak{A}}}$.

Conclusions:

- $\overline{\mathfrak{A}}$ is a $\sigma$-algebra, see (iv), (vi) and Theorem 1.1.(ii),
- $\mathfrak{A} \subset \overline{\mathfrak{A}}$ by (ii), hence $\sigma(\mathfrak{A}) \subset \overline{\mathfrak{A}}$.
- $\overline{\mu}|_{\overline{\mathfrak{A}}}$ is a measure with $\overline{\mu}|_{\mathfrak{A}} = \mu$, see (vi) and (i).

Put $\mu^* = \overline{\mu}|_{\sigma(\mathfrak{A})}$. □

**Remark 2.** The extension from Theorem 3 is non-unique, in general. For instance, put $\Omega = \mathbb{R}$ and

$$f(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ \infty & \text{otherwise} \end{cases}, \qquad A \subset \mathbb{R}.$$

Then $\mu = f|_{\mathfrak{A}}$ defines a pre-measure on the semi-algebra $\mathfrak{A} = \mathfrak{J}_1$ of intervals. Now we have

(i) a unique extension of $\mu$ to a pre-measure $\widehat{\mu}$ on $\mathfrak{A}^+$, namely $\widehat{\mu} = f|_{\mathfrak{A}^+}$,

(ii) the outer measure $\overline{\mu} = f$,

(iii) $\sigma(\mathfrak{A}) = \sigma(\mathfrak{A}^+) = \mathfrak{B}$.

For the counting measure $\mu_1$ on $\mathfrak{B}$ and for the measure $\mu_2 = f|_{\mathfrak{B}}$ according to the proof of Theorem 3 we have

$$\mu_1 \neq \mu_2 \wedge \mu_1|_{\mathfrak{A}^+} = \mu_2|_{\mathfrak{A}^+}.$$

**Definition 3.** $\mu : \mathfrak{A} \to \mathbb{R}_+ \cup \{\infty\}$ is called

(i) *σ-finite*, if

$$\exists\, B_1, B_2, \ldots \in \mathfrak{A} \text{ pairwise disjoint} : \quad \Omega = \bigcup_{i=1}^{\infty} B_i \wedge \forall\, i \in \mathbb{N} : \mu(B_i) < \infty,$$

(ii) *finite*, if $\Omega \in \mathfrak{A}$ and $\mu(\Omega) < \infty$.

**Theorem 4** (Uniqueness)**.** For measures $\mu_1$, $\mu_2$ on $\mathfrak{A}$ and $\mathfrak{A}_0 \subset \mathfrak{A}$ with

(i) $\sigma(\mathfrak{A}_0) = \mathfrak{A}$ and $\mathfrak{A}_0$ is closed w.r.t. intersections,

(ii) $\mu_1|_{\mathfrak{A}_0}$ is $\sigma$-finite,

(iii) $\mu_1|_{\mathfrak{A}_0} = \mu_2|_{\mathfrak{A}_0}$

we have

$$\mu_1 = \mu_2.$$

*Proof.* Take $B_i$ according to Definition 3, with $\mathfrak{A}_0$ instead of $\mathfrak{A}$, and put

$$\mathfrak{D}_i = \{A \in \mathfrak{A} : \mu_1(A \cap B_i) = \mu_2(A \cap B_i)\}.$$

Obviously, $\mathfrak{D}_i$ is a Dynkin class and $\mathfrak{A}_0 \subset \mathfrak{D}_i$. Theorem 1.2.(i) yields

$$\mathfrak{D}_i \subset \mathfrak{A} = \sigma(\mathfrak{A}_0) = \delta(\mathfrak{A}_0) \subset \mathfrak{D}_i.$$

Thus $\mathfrak{A} = \mathfrak{D}_i$ and for $A \in \mathfrak{A}$,

$$\mu_1(A) = \sum_{i=1}^{\infty} \mu_1(A \cap B_i) = \sum_{i=1}^{\infty} \mu_2(A \cap B_i) = \mu_2(A).$$

□

**Corollary 1.** For every semi-algebra $\mathfrak{A}$ and every pre-measure $\mu$ on $\mathfrak{A}$ that is $\sigma$-finite

$$\underset{1}{\exists}\, \mu^* \text{ measure on } \sigma(\mathfrak{A}) : \quad \mu^*|_{\mathfrak{A}} = \mu.$$

*Proof.* Use Theorems 2, 3, and 4. □

**Remark 3.** Applications of Corollary 1:

(i) For $\Omega = \mathbb{R}^k$ and the Lebesgue pre-measure $\lambda_k$ on $\mathfrak{I}_k$ we get the Lebesgue measure on $\mathfrak{B}_k$. Notation for the latter: $\lambda_k$.

(ii) In Example 1.(v) there exists a uniquely determined probability measure $P$ on $\bigotimes_{i=1}^{\infty} \mathfrak{P}(\{0,1\})$ such that

$$P(A_1 \times \cdots \times A_n \times \{0,1\} \times \ldots) = \frac{|A_1 \times \cdots \times A_n|}{|\{0,1\}^n|}$$

for $A_1, \ldots, A_n \subset \{0,1\}$. We will study the general construction of product measures in Section 8.

For a pre-measure $\mu$ on an algebra $\mathfrak{A}$ the Carathéodory construction yields the extensions

$$\big(\Omega,\ \sigma(\mathfrak{A}),\ \overline{\mu}|_{\sigma(\mathfrak{A})}\big), \qquad \big(\Omega,\ \overline{\mathfrak{A}}_{\overline{\mu}},\ \overline{\mu}|_{\overline{\mathfrak{A}}_{\overline{\mu}}}\big). \tag{2}$$

To what extend is $\overline{\mathfrak{A}}_{\overline{\mu}}$ larger than $\sigma(\mathfrak{A})$?

**Definition 4.** A measure space $(\Omega, \mathfrak{A}, \mu)$ is *complete* if

$$\mathfrak{N}_\mu \subset \mathfrak{A}$$

for

$$\mathfrak{N}_\mu = \{B \in \mathfrak{P}(\Omega) : \exists\, A \in \mathfrak{A} : B \subset A \wedge \mu(A) = 0\}.$$

**Theorem 5.** For a measure space $(\Omega, \mathfrak{A}, \mu)$ define

$$\mathfrak{A}^\mu = \{A \cup N : A \in \mathfrak{A}\ ,\ N \in \mathfrak{N}_\mu\}$$

and

$$\widetilde{\mu}(A \cup N) = \mu(A), \qquad A \in \mathfrak{A},\ N \in \mathfrak{N}_\mu.$$

Then

(i) $\widetilde{\mu}$ is well defined and $(\Omega, \mathfrak{A}^{\mu}, \widetilde{\mu})$ is a complete measure space with $\widetilde{\mu}|_{\mathfrak{A}} = \mu$, called the *completion of* $(\Omega, \mathfrak{A}, \mu)$,

(ii) for every complete measure space $(\Omega, \check{\mathfrak{A}}, \check{\mu})$ with $\check{\mathfrak{A}} \supset \mathfrak{A}$ and $\check{\mu}|_{\mathfrak{A}} = \mu$ we have $\check{\mathfrak{A}} \supset \mathfrak{A}^{\mu}$ and $\check{\mu}|_{\mathfrak{A}^{\mu}} = \widetilde{\mu}$.

*Proof.* See Gänssler, Stute (1977, p. 34) or Elstrodt (1996, p. 64). □

**Remark 4.** It is easy to verify that $\big(\Omega,\ \overline{\mathfrak{A}}_{\overline{\mu}},\ \overline{\mu}|_{\overline{\mathfrak{A}}_{\overline{\mu}}}\big)$ in (2) is complete. However, $\big(\Omega,\ \sigma(\mathfrak{A}),\ \overline{\mu}|_{\sigma(\mathfrak{A})}\big)$ is not complete in general, see Example 3 below.

**Theorem 6.** If $\mu$ is a $\sigma$-finite pre-measure on an algebra $\mathfrak{A}$, then $\big(\Omega,\ \overline{\mathfrak{A}}_{\overline{\mu}},\ \overline{\mu}|_{\overline{\mathfrak{A}}_{\overline{\mu}}}\big)$ is the completion of $\big(\Omega,\ \sigma(\mathfrak{A}),\ \overline{\mu}|_{\sigma(\mathfrak{A})}\big)$.

*Proof.* See Elstrodt (1996, p. 64). □

**Example 3.** Consider the completion $\big(\mathbb{R}^k, \mathfrak{L}_k, \widetilde{\lambda}_k\big)$ of $\big(\mathbb{R}^k, \mathfrak{B}_k, \lambda_k\big)$. Here $\mathfrak{L}_k$ is called the $\sigma$-algebra of *Lebesgue measurable sets* and $\widetilde{\lambda}_k$ is called the Lebesgue measure on $\mathfrak{L}_k$. Notation: $\lambda_k = \widetilde{\lambda}_k$. We have

$$\mathfrak{B}_k \subsetneq \mathfrak{L}_k,$$

hence $(\mathbb{R}^k, \mathfrak{B}_k, \lambda_k)$ is not complete.
Proof: Assume $k = 1$ for simplicity. For the Cantor set $C \subset \mathbb{R}$

$$C \in \mathfrak{B}_1 \wedge \lambda_1(C) = 0 \wedge |C| = |\mathbb{R}|.$$

By Theorem 3.4, $|\mathfrak{B}_1| = |\mathbb{R}|$, but

$$|\{0,1\}^{\mathbb{R}}| = |\mathfrak{P}(C)| \leq |\mathfrak{L}_k| \leq |\{0,1\}^{\mathbb{R}}|.$$

We add that $\mathfrak{L}_k \subsetneq \mathfrak{P}(\mathbb{R}^k)$, see Elstrodt (1996, §III.3).

# 5 Integration

For the proofs, see Analysis IV or Elstrodt (1996, Kap. VI).
Given: a measure space $(\Omega, \mathfrak{A}, \mu)$. Notation: $\mathfrak{S}_+ = \mathfrak{S}_+(\Omega, \mathfrak{A})$ is the class of non-negative simple functions.

**Definition 1.** *Integral* of $f \in \mathfrak{S}_+$ w.r.t. $\mu$

$$\int f \, d\mu = \sum_{i=1}^{n} \alpha_i \cdot \mu(A_i)$$

if

$$f = \sum_{i=1}^{n} \alpha_i \cdot 1_{A_i}$$

with $\alpha_i \geq 0$ and $A_i \in \mathfrak{A}$. (Note that the integral is well defined.)

**Lemma 1.** For $f, g \in \mathfrak{S}_+$ and $c \in \mathbb{R}_+$

(i) $\int (f+g)\, d\mu = \int f\, d\mu + \int g\, d\mu$,

(ii) $\int (cf)\, d\mu = c \cdot \int f\, d\mu$,

(iii) $f \le g \Rightarrow \int f\, d\mu \le \int g\, d\mu$ (*monotonicity*).

Notation: $\overline{\mathfrak{Z}}_+ = \overline{\mathfrak{Z}}_+(\Omega, \mathfrak{A})$ is the class of nonnegative $\mathfrak{A}$-$\overline{\mathfrak{B}}$-measurable functions.

**Definition 2.** *Integral* of $f \in \overline{\mathfrak{Z}}_+$ w.r.t. $\mu$

$$\int f\, d\mu = \sup\Big\{ \int g\, d\mu : g \in \mathfrak{S}_+ \wedge g \le f \Big\}.$$

**Theorem 1** (Monotone convergence, Beppo Levi)**.** Let $f_n \in \overline{\mathfrak{Z}}_+$ such that

$$\forall\, n \in \mathbb{N} : f_n \le f_{n+1}.$$

Then

$$\int \sup_n f_n\, d\mu = \sup_n \int f_n\, d\mu.$$

**Remark 1.** For every $f \in \overline{\mathfrak{Z}}_+$ there exists a sequence of functions $f_n \in \mathfrak{S}_+$ such that $f_n \uparrow f$, see Theorem 2.7.

**Example 1.** Consider

$$f_n = \frac{1}{n} \cdot 1_{[0,n]}$$

on $(\mathbb{R}, \mathfrak{B}, \lambda_1)$. Then

$$\int f_n\, d\lambda_1 = 1, \qquad \lim_{n\to\infty} f_n = 0.$$

**Lemma 2.** The conclusions from Lemma 1 remain valid on $\overline{\mathfrak{Z}}_+$.

**Theorem 2** (Fatou's Lemma)**.** For every sequence $(f_n)_n$ in $\overline{\mathfrak{Z}}_+$

$$\int \liminf_{n\to\infty} f_n\, d\mu \le \liminf_{n\to\infty} \int f_n\, d\mu.$$

*Proof.* For $g_n = \inf_{k \ge n} f_k$ we have $g_n \in \overline{\mathfrak{Z}}_+$ and $g_n \uparrow \liminf_n f_n$. By Theorem 1 and Lemma 1.(iii)

$$\int \liminf_n f_n\, d\mu = \lim_{n\to\infty} \int g_n\, d\mu \le \liminf_{n\to\infty} \int f_n\, d\mu.$$

□

**Theorem 3.** Let $f \in \overline{\mathfrak{Z}}_+$. Then

$$\int f\, d\mu = 0 \Leftrightarrow \mu(\{f > 0\}) = 0.$$

**Definition 3.** A property $\Pi$ holds *$\mu$-almost everywhere ($\mu$-a.e., a.e.)*, if

$$\exists\, A \in \mathfrak{A} : \{\omega \in \Omega : \Pi \text{ does not hold for } \omega\} \subset A \wedge \mu(A) = 0.$$

In case of a probability measure we say: *$\mu$-almost surely, $\mu$-a.s., with probability one.*

Notation: $\overline{\mathfrak{Z}} = \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$ is the class of $\mathfrak{A}$-$\overline{\mathfrak{B}}$-measurable functions.

**Definition 4.** $f \in \overline{\mathfrak{Z}}$ *quasi-$\mu$-integrable* if

$$\int f_+ \, d\mu < \infty \quad \vee \quad \int f_- \, d\mu < \infty.$$

In this case: *integral* of $f$ (w.r.t. $\mu$)

$$\int f \, d\mu = \int f_+ \, d\mu - \int f_- \, d\mu.$$

$f \in \overline{\mathfrak{Z}}$ *$\mu$-integrable* if

$$\int f_+ \, d\mu < \infty \quad \wedge \quad \int f_- \, d\mu < \infty.$$

**Theorem 4.**

(i) $f$ $\mu$-integrable $\Rightarrow \mu(\{|f| = \infty\}) = 0$,

(ii) $f$ $\mu$-integrable $\wedge\ g \in \overline{\mathfrak{Z}} \wedge f = g$ $\mu$-a.e. $\Rightarrow g$ $\mu$-integrable $\wedge \int f \, d\mu = \int g \, d\mu$.

(iii) equivalent properties for $f \in \overline{\mathfrak{Z}}$:

- (a) $f$ $\mu$-integrable,
- (b) $|f|$ $\mu$-integrable,
- (c) $\exists\, g : g$ $\mu$-integrable $\wedge\ |f| \le g$ $\mu$-a.e.,

(iv) for $f$ and $g$ $\mu$-integrable and $c \in \mathbb{R}$

- (a) $f+g$ well-defined $\mu$-a.e. and $\mu$-integrable with $\int (f+g) \, d\mu = \int f \, d\mu + \int g \, d\mu$,
- (b) $c \cdot f$ $\mu$-integrable with $\int (cf) \, d\mu = c \cdot \int f \, d\mu$,
- (c) $f \le g$ $\mu$-a.e. $\Rightarrow \int f \, d\mu \le \int g \, d\mu$.

**Remark 2.** An outlook. Consider an arbitrary set $\Omega \ne \emptyset$ and a vector space $\mathfrak{F} \subset \mathbb{R}^\Omega$ such that

$$f \in \mathfrak{F} \Rightarrow \big(|f| \in \mathfrak{F} \wedge \inf\{f, 1\} \in \mathfrak{F}\big).$$

A monotone linear mapping $I : \mathfrak{F} \to \mathbb{R}$ such that

$$f, f_1, f_2, \ldots \in \mathfrak{F} \wedge f_n \uparrow f \Rightarrow I(f) = \lim_{n\to\infty} I(f_n)$$

is called an *abstract integral*. Note that

$$I(f) = \int f \, d\mu$$

defines an abstract integral on

$$\mathfrak{F} = \{f \in \mathfrak{Z}(\Omega, \mathfrak{A}) : f \ \mu\text{-integrable}\} = \mathfrak{L}^1(\Omega, \mathfrak{A}, \mu).$$

*Daniell-Stone-Theorem:* for every abstract integral there exists a uniquely determined measure $\mu$ on $\mathfrak{A} = \sigma(\mathfrak{F})$ such that

$$\mathfrak{F} \subset \mathfrak{L}^1(\Omega, \mathfrak{A}, \mu) \wedge \forall f \in \mathfrak{F} : I(f) = \int f \, d\mu.$$

See Bauer (1978, Satz 39.4) or Floret (1981).

Application: *Riesz representation theorem.* Here $\mathfrak{F} = C([0,1])$ and $I : \mathfrak{F} \to \mathbb{R}$ linear and monotone. Then $I$ is an abstract integral, which follows from Dini's Theorem, see Floret (1981, p. 45). Hence there exists a uniquely determined measure $\mu$ on $\sigma(\mathfrak{F}) = \mathfrak{B}([0,1])$ such that

$$\forall f \in \mathfrak{F} : I(f) = \int f \, d\mu.$$

**Theorem 5** (Dominated convergence, Lebesgue)**.** Assume that

(i) $f_n \in \overline{\mathfrak{Z}}$ for $n \in \mathbb{N}$,

(ii) $\exists \, g$ $\mu$-integrable $\forall n \in \mathbb{N} : |f_n| \le g$ $\mu$-a.e.,

(iii) $f \in \overline{\mathfrak{Z}}$ such that $\lim_{n\to\infty} f_n = f$ $\mu$-a.e.

Then $f$ is $\mu$-integrable and

$$\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu.$$

**Example 2.** Consider

$$f_n = n \cdot 1_{]0,1/n[}$$

on $(\mathbb{R}, \mathfrak{B}, \lambda_1)$. Then

$$\int f_n \, d\lambda_1 = 1, \qquad \lim_{n\to\infty} f_n = 0.$$

# 6 $\mathfrak{L}^p$-Spaces

Given: a measure space $(\Omega, \mathfrak{A}, \mu)$ and $1 \le p < \infty$. Put $\mathfrak{Z} = \mathfrak{Z}(\Omega, \mathfrak{A})$.

**Definition 1.**

$$\mathfrak{L}^p = \mathfrak{L}^p(\Omega, \mathfrak{A}, \mu) = \Big\{ f \in \mathfrak{Z} : \int |f|^p \, d\mu < \infty \Big\}.$$

In particular, for $p = 1$: *integrable functions* and $\mathfrak{L} = \mathfrak{L}^1$, and for $p = 2$: *square-integrable functions*. Put

$$\|f\|_p = \left( \int |f|^p \, d\mu \right)^{1/p}, \qquad f \in \mathfrak{L}^p.$$

**Theorem 1** (Hölder inequality)**.** Let $1 < p, q < \infty$ such that $1/p + 1/q = 1$ and let $f \in \mathfrak{L}^p$, $g \in \mathfrak{L}^q$. Then
$$\int |f \cdot g| \, d\mu \le \|f\|_p \cdot \|g\|_q.$$
In particular, for $p = q = 2$: *Cauchy-Schwarz inequality.*

*Proof.* See Analysis IV or Elstrodt (1996, §VI.1) as well as Theorem 5.3. □

**Theorem 2.** $\mathfrak{L}^p$ is a vector space and $\|\cdot\|_p$ is a semi-norm on $\mathfrak{L}^p$. Furthermore,
$$\|f\|_p = 0 \quad \Leftrightarrow \quad f = 0 \ \mu\text{-a.e.}$$

*Proof.* See Analysis IV or Elstrodt (1996, §VI.2). □

**Definition 2.** Let $f, f_n \in \mathfrak{L}^p$ for $n \in \mathbb{N}$. $(f_n)_n$ *converges to* $f$ *in* $\mathfrak{L}^p$ *(in mean of order* $p$*)* if
$$\lim_{n\to\infty} \|f - f_n\|_p = 0.$$
In particular, for $p = 1$: *convergence in mean*, and for $p = 2$: *mean-square convergence*. Notation:
$$f_n \xrightarrow{\mathfrak{L}^p} f.$$

**Remark 1.** Let $f, f_n \in \overline{\mathfrak{Z}}$ for $n \in \mathbb{N}$. Recall (define) that $(f_n)_n$ *converges to* $f$ $\mu$*-a.e.* if
$$\mu(A^c) = 0$$
for
$$A = \left\{\lim_{n\to\infty} f_n = f\right\} = \left\{\limsup_{n\to\infty} f_n = \liminf_{n\to\infty} f_n\right\} \cap \left\{\limsup_{n\to\infty} f_n = f\right\} \in \mathfrak{A}.$$
Notation:
$$f_n \xrightarrow{\mu\text{-a.e.}} f.$$

**Lemma 1.** Let $f, g, f_n \in \mathfrak{L}^p$ for $n \in \mathbb{N}$ such that $f_n \xrightarrow{\mathfrak{L}^p} f$. Then
$$f_n \xrightarrow{\mathfrak{L}^p} g \quad \Leftrightarrow \quad f = g \ \mu\text{-a.e.}$$
Analogously for convergence almost everywhere.

*Proof.* For convergence in $\mathfrak{L}^p$: '$\Leftarrow$' follows from Theorem 5.4.(ii). Use
$$\|f - g\|_p \le \|f - f_n\|_p + \|f_n - g\|_p$$
to verify '$\Rightarrow$'.
For convergence almost everywhere: '$\Leftarrow$' trivially holds. Use
$$\left\{\lim_{n\to\infty} f_n = f\right\} \cap \left\{\lim_{n\to\infty} f_n = g\right\} \subset \{f = g\}$$
to verify '$\Rightarrow$'. □

**Theorem 3** (Fischer-Riesz)**.** Consider a sequence $(f_n)_n$ in $\mathfrak{L}^p$. Then

(i) $(f_n)_n$ Cauchy sequence $\Rightarrow \exists\, f \in \mathfrak{L}^p :\ f_n \xrightarrow{\mathfrak{L}^p} f$ (*completeness*),

(ii) $f_n \xrightarrow{\mathfrak{L}^p} f \Rightarrow \exists$ subsequence $(f_{n_k})_k : f_{n_k} \xrightarrow{\mu\text{-a.e.}} f$.

*Proof.* Ad (i): Consider a Cauchy sequence $(f_n)_n$ and a subsequence $(f_{n_k})_k$ such that

$$\forall\, k \in \mathbb{N}\ \forall\, m \geq n_k : \|f_m - f_{n_k}\|_p \leq 2^{-k}.$$

For

$$g_k = f_{n_{k+1}} - f_{n_k} \in \mathfrak{L}^p$$

we have

$$\Big\|\sum_{\ell=1}^{k} |g_\ell|\Big\|_p \leq \sum_{\ell=1}^{k} \|g_\ell\|_p \leq \sum_{\ell=1}^{k} 2^{-\ell} \leq 1.$$

Put $g = \sum_{\ell=1}^{\infty} |g_\ell| \in \overline{\mathfrak{Z}}_+$. By Theorem 5.1

$$\int g^p \, d\mu = \int \sup_k \Big(\sum_{\ell=1}^{k} |g_\ell|\Big)^p d\mu = \sup_k \int \Big(\sum_{\ell=1}^{k} |g_\ell|\Big)^p d\mu \leq 1. \tag{1}$$

Thus, in particular, $\sum_{\ell=1}^{\infty} |g_\ell|$ and $\sum_{\ell=1}^{\infty} g_\ell$ converge $\mu$-a.e., see Theorem 5.4.(i). Since

$$f_{n_{k+1}} = \sum_{\ell=1}^{k} g_\ell + f_{n_1},$$

we have

$$f = \lim_{k\to\infty} f_{n_k}\ \mu\text{-a.e.}$$

for some $f \in \mathfrak{Z}$. Furthermore,

$$|f - f_{n_k}| \leq \sum_{\ell=k}^{\infty} |g_\ell| \leq g\ \mu\text{-a.e.},$$

so that, by Theorem 5.5 and (1),

$$\lim_{k\to\infty} \int |f - f_{n_k}|^p \, d\mu = 0.$$

It follows that

$$\lim_{n\to\infty} \|f - f_n\|_p = 0,$$

too. Finally, by Theorem 2, $f \in \mathfrak{L}^p$.
Ad (ii): Assume that

$$f_n \xrightarrow{\mathfrak{L}^p} f.$$

According to the proof of (i) there exists $\widetilde{f} \in \mathfrak{L}^p$ and a subsequence $(f_{n_k})_k$ such that

$$f_{n_k} \xrightarrow{\mu\text{-a.e.}} \widetilde{f} \wedge f_{n_k} \xrightarrow{\mathfrak{L}^p} \widetilde{f}.$$

Use Lemma 1. □

**Example 1.** Let $(\Omega, \mathfrak{A}, \mu) = \big([0,1], \mathfrak{B}([0,1]), \lambda_1|_{\mathfrak{B}([0,1])}\big)$. (By Remark 1.7.(ii) we have $\mathfrak{B}([0,1]) \subset \mathfrak{B}_1$). Define

$$A_1 = [0,1]$$
$$A_2 = [0,1/2], \quad A_3 = [1/2,1]$$
$$A_4 = [0,1/3], \quad A_5 = [1/3,2/3], \quad A_6 = [2/3,1]$$
$$\text{etc.}$$

Put $f_n = 1_{A_n}$. Then

$$\lim_{n\to\infty} \|f_n - 0\|_p = \lim_{n\to\infty} \|f_n\|_p = 0 \tag{2}$$

but

$$\{(f_n)_n \text{ converges}\} = \emptyset.$$

**Remark 2.** Define

$$\mathfrak{L}^\infty = \mathfrak{L}^\infty(\Omega, \mathfrak{A}, P) = \{f \in \mathfrak{Z} : \exists\, c \in \mathbb{R}_+ : |f| \le c \ \mu\text{-a.e.}\}$$

and

$$\|f\|_\infty = \inf\{c \in \mathbb{R}_+ : |f| \le c \ \mu\text{-a.e.}\}, \qquad f \in \mathfrak{L}^\infty.$$

$f \in \mathfrak{L}^\infty$ is called *essentially bounded* and $\|f\|_\infty$ is called the *essential supremum of* $|f|$. Use Theorem 4.1.(iii) to verify that

$$|f| \le \|f\|_\infty \ \mu\text{-a.e.}$$

The definitions and results of this section, except (2), extend to the case $p = \infty$, where $q = 1$ in Theorem 1. In Theorem 3.(ii) we even have $f_n \xrightarrow{\mathfrak{L}^\infty} f \Rightarrow f_n \xrightarrow{\mu\text{-a.e.}} f$.

**Remark 3.** Put

$$\mathfrak{N}^p = \{f \in \mathfrak{L}^p : f = 0 \ \mu\text{-a.e.}\}$$

Then the quotient space $L^p = \mathfrak{L}^p/\mathfrak{N}^p$ is a Banach space. In particular, for $p = 2$, $L^2$ is a Hilbert space, with semi-inner product on $\mathfrak{L}^2$ given by

$$\langle f, g\rangle = \int f \cdot g \, d\mu, \qquad f, g \in \mathfrak{L}^2.$$

**Theorem 4.** If $\mu$ is finite and $1 \le p < q \le \infty$ then

$$\mathfrak{L}^q \subset \mathfrak{L}^p$$

and

$$\|f\|_p \le \mu(\Omega)^{1/p-1/q} \cdot \|f\|_q, \qquad f \in \mathfrak{L}^q.$$

*Proof.* The result trivially holds for $q = \infty$.In the sequel, $q < \infty$. Use $|f|^p \le 1 + |f|^q$ and Theorem 5.4.(iii) to obtain $\mathfrak{L}^q \subset \mathfrak{L}^p$. Put $r = q/p$ and define $s$ by $1/r + 1/s = 1$. Theorem 1 yields

$$\int |f|^p \, d\mu \le \left(\int |f|^{p\cdot r} \, d\mu\right)^{1/r} \cdot \big(\mu(\Omega)\big)^{1/s}.$$

□

**Example 2.** Let $1 \le p < q \le \infty$. With respect to the counting measure on $\mathfrak{P}(\mathbb{N})$, $\mathfrak{L}^p \subset \mathfrak{L}^q$. With respect to the Lebesgue measure on $\mathfrak{B}_k$ neither $\mathfrak{L}^q \subset \mathfrak{L}^p$ nor $\mathfrak{L}^p \subset \mathfrak{L}^q$.