Probability Theory

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Literature

In particular,

H. Bauer, *Probability Theory*, de Gruyter, Berlin, 1996.

P. Billingsley, *Probability and Measure*, Wiley, New York, first edition 1979, third edition 1995.

J. Elstrodt, *Maß- und Integrationstheorie*, Springer, Berlin, first edition 1996, fifth edition 2007.

P. Gänssler, W. Stute, *Wahrscheinlichkeitstheorie*, Springer, Berlin, 1977.

A. Klenke, *Wahrscheinlichkeitstheorie*, Springer, Berlin, first edition 2006, second edition 2008.

Contents

Chapter I

Introduction

A *stochastic model*: a probability space $(\Omega, \mathfrak{A}, P)$ together with a collection of random variables (measurable mappings) $\Omega \to \mathbb{R}$, say.

Examples of probability spaces, known from 'Introduction to Stochastics' or 'Analysis':

(i) Given: a countable set Ω and $f : \Omega \to \mathbb{R}_+$ such that $\sum_{\omega \in \Omega} f(\omega) = 1$.
Take the power set $\mathfrak{A} = \mathfrak{P}(\Omega)$ and define

$$P(A) = \sum_{\omega \in A} f(\omega), \qquad A \subset \Omega.$$

(ii) Given: $f : \mathbb{R}^k \to \mathbb{R}_+$ such that $\int_{\mathbb{R}^k} f(\omega)\, d\omega = 1$.
Let $\Omega = \mathbb{R}^k$, take the σ-algebra $\mathfrak{A} = \mathfrak{B}(\mathbb{R}^k)$ of Borel sets in $\mathbb{R}^k$ and define

$$P(A) = \int_A f(\omega)\, d\omega, \qquad A \in \mathfrak{B}(\mathbb{R}^k).$$

Main topics in this course:

(i) construction of probability spaces, including the theory of measure and integration,

(ii) limit theorems,

(iii) conditional probabilities and expectations,

(iv) discrete-time martingales.

Example 1. Limit theorems like the law of large numbers or the central limit theorem deal with sequences $X_1, X_2, \dots$ of random variables and their partial sums

$$S_n = \sum_{i=1}^{n} X_i$$

(gambling: cumulative gain after n trials; physics: position of a particle after n collisions).

Under which conditions and in which sense does S_n/n or $S_n/\sqrt{n}$ converge, as n tends to infinity?

Example 2. Limit theorems hold in particular for independent and identically distributed (i.i.d.) random variables $X_1, X_2, \ldots$ with $\mathrm{E}(X_i) = 0$ and $\mathrm{Var}(X_i) = 1$. Then S_n/n 'converges' to zero and $S_n/\sqrt{n}$ 'converges' to the standard normal distribution. In particular, in a simple case of gambling: X_i takes values ± 1 with probability $1/2$. Existence of such a model? Existence for every choice of the distribution of X_i?

Example 3. The fluctuation of a stock price defines a function on the time interval $\mathbb{R}_+$ with values in $\mathbb{R}$ (for simplicity, we admit negative stock prices at this point). What is a reasonable σ-algebra on the space Ω of all mappings $\mathbb{R}_+ \to \mathbb{R}$ or on the subspace of all continuous mappings? How can we define (non-discrete) probability measures on these spaces in order to model the random dynamics of stock prices? Analogously for random perturbations in physics, biology, etc.

More generally, the same questions arise for mappings $I \to S$ with an arbitrary non-empty set I and $S \subset \mathbb{R}^d$ (physics: phase transition in ferromagnetic materials, the orientation of magnetic dipoles on a set I of sites; medicine: spread of diseases, certain biometric parameters for a set I of individuals; environmental science: the concentration of certain pollutants in a region I).

Example 4. Consider two random variables X_1 and X_2. If $P(\{X_2 = v\}) > 0$ then the conditional probability of $\{X_1 \in A\}$ given $\{X_2 = v\}$ is defined by

$$P(\{X_1 \in A\} \mid \{X_2 = v\}) = \frac{P(\{X_1 \in A\} \cap \{X_2 = v\})}{P(\{X_2 = v\})}.$$

How can we reasonably extend this definition to the case $P(\{X_2 = v\}) = 0$, e.g., for X_2 being normally distributed? How does the observation $X_2 = v$ change our stochastic model? Cf. Example 3.

Chapter II

Measure and Integral

1 Classes of Sets

Given: a non-empty set Ω and a class $\mathfrak{A} \subset \mathfrak{P}(\Omega)$ of subsets. Put

$$\mathfrak{A}^+ = \left\{ \bigcup_{i=1}^{n} A_i : n \in \mathbb{N} \wedge A_1, \dots, A_n \in \mathfrak{A} \text{ pairwise disjoint} \right\}.$$

Definition 1.

(i) $\mathfrak{A}$ *closed w.r.t. intersections* if $A, B \in \mathfrak{A} \Rightarrow A \cap B \in \mathfrak{A}$.

(ii) $\mathfrak{A}$ *closed w.r.t. unions* if $A, B \in \mathfrak{A} \Rightarrow A \cup B \in \mathfrak{A}$.

(iii) $\mathfrak{A}$ *semi-algebra (in Ω)* if

  (a) $\Omega \in \mathfrak{A}$,

  (b) $\mathfrak{A}$ closed w.r.t. intersections,

  (c) $A \in \mathfrak{A} \Rightarrow A^c \in \mathfrak{A}^+$.

(iv) $\mathfrak{A}$ *algebra (in Ω)* if

  (a) $\Omega \in \mathfrak{A}$,

  (b) $\mathfrak{A}$ closed w.r.t. intersections,

  (c) $A \in \mathfrak{A} \Rightarrow A^c \in \mathfrak{A}$.

(v) $\mathfrak{A}$ *σ-algebra (in Ω)* if

  (a) $\Omega \in \mathfrak{A}$,

  (b) $A_1, A_2, \dots \in \mathfrak{A} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathfrak{A}$,

  (c) $A \in \mathfrak{A} \Rightarrow A^c \in \mathfrak{A}$.

Remark 1. Let $\mathfrak{A}$ denote a σ-algebra in Ω. Recall that a *probability measure* P on $(\Omega, \mathfrak{A})$ is a mapping

$$P : \mathfrak{A} \to [0, 1]$$

such that $P(\Omega) = 1$ and

$$A_1, A_2, \ldots \in \mathfrak{A} \text{ pairwise disjoint} \quad \Rightarrow \quad P\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} P(A_i).$$

Moreover, $(\Omega, \mathfrak{A}, P)$ is called a *probability space*, and $P(A)$ is the *probability* of the *event* $A \in \mathfrak{A}$.

Remark 2.

(i) $\mathfrak{A}$ σ-algebra $\Rightarrow$ $\mathfrak{A}$ algebra $\Rightarrow$ $\mathfrak{A}$ semi-algebra.

(ii) $\mathfrak{A}$ closed w.r.t. intersections $\Rightarrow$ $\mathfrak{A}^+$ closed w.r.t. intersections.

(iii) $\mathfrak{A}$ algebra and $A_1, A_2 \in \mathfrak{A} \Rightarrow A_1 \cup A_2,\ A_1 \setminus A_2,\ A_1 \bigtriangleup A_2 \in \mathfrak{A}$.

(iv) $\mathfrak{A}$ σ-algebra and $A_1, A_2, \ldots \in \mathfrak{A} \Rightarrow \bigcap_{n=1}^{\infty} A_n \in \mathfrak{A}$.

Example 1.

(i) Let $\Omega = \mathbb{R}$ and consider the class of intervals

$$\mathfrak{A} = \{]a, b] : a, b \in \mathbb{R} \wedge a < b\} \cup \{]-\infty, b] : b \in \mathbb{R}\} \cup \{]a, \infty[: a \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}.$$

Then $\mathfrak{A}$ is a semi-algebra, but not an algebra.

(ii) $\{A \in \mathfrak{P}(\Omega) : A \text{ finite or } A^c \text{ finite}\}$ is an algebra, but not a σ-algebra in general.

(iii) $\{A \in \mathfrak{P}(\Omega) : A \text{ countable or } A^c \text{ countable}\}$ is a σ-algebra.

(iv) $\mathfrak{P}(\Omega)$ is the largest σ-algebra in Ω, $\{\emptyset, \Omega\}$ is the smallest σ-algebra in Ω.

Definition 2. $\mathfrak{A}$ *Dynkin class (in Ω)* if

(i) $\Omega \in \mathfrak{A}$,

(ii) $A_1, A_2 \in \mathfrak{A} \wedge A_1 \subset A_2 \Rightarrow A_2 \setminus A_1 \in \mathfrak{A}$,

(iii) $A_1, A_2, \ldots \in \mathfrak{A}$ pairwise disjoint $\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathfrak{A}$.

Remark 3. $\mathfrak{A}$ σ-algebra $\Rightarrow$ $\mathfrak{A}$ Dynkin class.

Theorem 1. For every Dynkin class $\mathfrak{A}$

$$\mathfrak{A} \ \sigma\text{-algebra} \quad \Leftrightarrow \quad \mathfrak{A} \text{ closed w.r.t. intersections.}$$

Proof. '$\Leftarrow$': For $A \in \mathfrak{A}$ we have $A^c = \Omega \setminus A \in \mathfrak{A}$ since $\mathfrak{A}$ is a Dynkin class. For $A, B \in \mathfrak{A}$ we have

$$A \cup B = A \cup (B \setminus (A \cap B)) \in \mathfrak{A}$$

since $\mathfrak{A}$ is also closed w.r.t. intersections. Thus, for $A_1, A_2, \ldots \in \mathfrak{A}$ and $B_m = \bigcup_{n=1}^m A_n$ we get $B_m \in \mathfrak{A}$ and

$$\bigcup_{n=1}^\infty A_n = \bigcup_{m=1}^\infty (B_m \setminus B_{m-1}) \in \mathfrak{A},$$

where $B_0 = \emptyset$. □

Remark 4. Consider σ-algebras (algebras, Dynkin classes) $\mathfrak{A}_i$ for $i \in I \neq \emptyset$. Then $\bigcap_{i \in I} \mathfrak{A}_i$ is a σ-algebra (algebra, Dynkin class), too. See also Übung 1.2.

Given: a class $\mathfrak{E} \subset \mathfrak{P}(\Omega)$.

Definition 3. The *σ-algebra generated by* $\mathfrak{E}$

$$\sigma(\mathfrak{E}) = \bigcap \{\mathfrak{A} : \mathfrak{A} \ \sigma\text{-algebra in } \Omega \wedge \mathfrak{E} \subset \mathfrak{A}\}.$$

Analogously, $\alpha(\mathfrak{E})$, $\delta(\mathfrak{E})$ the *algebra, Dynkin class,* respectively, *generated by* $\mathfrak{E}$.

Remark 5. For $\gamma \in \{\sigma, \alpha, \delta\}$ and $\mathfrak{E}, \mathfrak{E}_1, \mathfrak{E}_2 \subset \mathfrak{P}(\Omega)$

(i) $\gamma(\mathfrak{E})$ is the smallest 'γ-class' that contains $\mathfrak{E}$,

(ii) $\mathfrak{E}_1 \subset \mathfrak{E}_2 \Rightarrow \gamma(\mathfrak{E}_1) \subset \gamma(\mathfrak{E}_2)$,

(iii) $\gamma(\gamma(\mathfrak{E})) = \gamma(\mathfrak{E})$.

Example 2. Let $\Omega = \mathbb{N}$ and $\mathfrak{E} = \{\{n\} : n \in \mathbb{N}\}$. Then

$$\alpha(\mathfrak{E}) = \{A \in \mathfrak{P}(\Omega) : A \text{ finite or } A^c \text{ finite}\} =: \mathfrak{A}.$$

Proof: $\mathfrak{A}$ is an algebra, see Example 1, and $\mathfrak{E} \subset \mathfrak{A}$. Thus $\alpha(\mathfrak{E}) \subset \mathfrak{A}$. On the other hand, for every finite set $A \subset \Omega$ we have $A = \bigcup_{n \in A} \{n\} \in \alpha(\mathfrak{E})$, and for every set $A \subset \Omega$ with finite complement we have $A = (A^c)^c \in \alpha(\mathfrak{E})$. Thus $\mathfrak{A} \subset \alpha(\mathfrak{E})$.
Moreover,

$$\sigma(\mathfrak{E}) = \mathfrak{P}(\mathbb{N}), \qquad \delta(\mathfrak{E}) = \mathfrak{P}(\mathbb{N}).$$

Theorem 2. $\mathfrak{E}$ closed w.r.t. intersections $\Rightarrow \sigma(\mathfrak{E}) = \delta(\mathfrak{E})$.

Proof. Remark 3 implies

$$\delta(\mathfrak{E}) \subset \sigma(\mathfrak{E}).$$

We claim that

$$\delta(\mathfrak{E}) \text{ is closed w.r.t. intersections.} \tag{1}$$

Then, by Theorem 1.(ii),

$$\sigma(\mathfrak{E}) \subset \delta(\mathfrak{E}).$$

Put

$$\mathfrak{C}_B = \{C \subset \Omega : C \cap B \in \delta(\mathfrak{E})\}, \qquad B \in \delta(\mathfrak{E}),$$

so that (1) is equivalent to

$$\forall\, B \in \delta(\mathfrak{E}) : \delta(\mathfrak{E}) \subset \mathfrak{C}_B. \tag{2}$$

It is straightforward to verify that

$$\forall\, B \in \delta(\mathfrak{E}) : \mathfrak{C}_B \text{ Dynkin class.} \tag{3}$$

Moreover, since $\mathfrak{E}$ is closed w.r.t. intersections,

$$\forall\, E \in \mathfrak{E} : \mathfrak{E} \subset \mathfrak{C}_E.$$

Therefore

$$\forall\, E \in \mathfrak{E} : \delta(\mathfrak{E}) \subset \mathfrak{C}_E,$$

which is equivalent to

$$\forall\, B \in \delta(\mathfrak{E}) : \mathfrak{E} \subset \mathfrak{C}_B.$$

Use (3) to obtain (2). □

An algebra $\alpha(\mathfrak{E})$ can be described explicitly, see Gänssler, Stute (1977, p. 14). The corresponding problem for σ-algebras is addressed in Billingsley (1979, p. 24). Here we only state the following fact.

Lemma 1. $\mathfrak{E}$ semi-algebra $\Rightarrow \alpha(\mathfrak{E}) = \mathfrak{E}^+$.

Proof. Clearly $\mathfrak{E} \subset \mathfrak{E}^+ \subset \alpha(\mathfrak{E})$. It remains to show that $\mathfrak{E}^+$ is an algebra. See Gänssler, Stute (1977, p. 14) for details. □

Sometimes it will be convenient to extend the reals as follows. Put

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\},$$

and define for every $a \in \mathbb{R}$

$$(\pm\infty) + (\pm\infty) = a + (\pm\infty) = (\pm\infty) + a = \pm\infty, \qquad a/_{\pm\infty} = 0,$$

$$a \cdot (\pm\infty) = (\pm\infty) \cdot a = \begin{cases} \pm\infty & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ \mp\infty & \text{if } a < 0 \end{cases}$$

as well as $-\infty < a < \infty$. For instance, the class $\mathfrak{A}$ from Example 1.(i) consists of the sets

$$\{x \in \mathbb{R} : a < x \leq b\}, \qquad a, b \in \overline{\mathbb{R}}.$$

Furthermore, $\lim_{n\to\infty} x_n = \pm\infty$ for a sequence $(x_n)_{n\in\mathbb{N}}$ in $\overline{\mathbb{R}}$ if for all $M \in \,]0, \infty[$ there is an integer n_0 such that $x_n \gtreqless \pm M$ for all $n \geq n_0$.

Recall that $(\Omega, \mathfrak{G})$ is a *topological space* if $\mathfrak{G} \subset \mathfrak{P}(\Omega)$ satisfies

(i) $\emptyset, \Omega \in \mathfrak{G}$,

(ii) $\mathfrak{G}$ is closed w.r.t. to intersections,

(iii) for every family $(G_i)_{i \in I}$ with $G_i \in \mathfrak{G}$ we have $\bigcup_{i \in I} G_i \in \mathfrak{G}$.

The elements $G \in \mathfrak{G}$ are called the *open subsets* of Ω, and their complements are called the *closed subsets* of Ω. A set $K \subset \Omega$ is called *compact* if for every family $(G_i)_{i \in I}$ with $G_i \in \mathfrak{G}$ and

$$K \subset \bigcup_{i \in I} G_i$$

there is a finite set $I_0 \subset I$ such that

$$K \subset \bigcup_{i \in I_0} G_i.$$

On $\Omega = \mathbb{R}^k$ and $\Omega = \overline{\mathbb{R}}^k$ we consider the natural topologies, and we use $\mathfrak{G}_k$ to denote the corresponding class of open sets in $\mathbb{R}^k$. In particular, $O \subset \overline{\mathbb{R}}$ is an open set iff $O \cap \mathbb{R} \in \mathfrak{G}_k$ and $]a, \infty] \subset O$ for some $a < \infty$ if $\infty \in O$ and $[-\infty, a[\subset O$ for some $a > -\infty$ if $-\infty \in O$.

Definition 4. For every topological space $(\Omega, \mathfrak{G})$

$$\mathfrak{B}(\Omega) = \sigma(\mathfrak{G})$$

is the *Borel-σ-algebra (in Ω w.r.t. $\mathfrak{G}$)*. In particular,

$$\mathfrak{B}_k = \mathfrak{B}(\mathbb{R}^k), \quad \mathfrak{B} = \mathfrak{B}_1, \quad \overline{\mathfrak{B}}_k = \mathfrak{B}(\overline{\mathbb{R}}^k), \quad \overline{\mathfrak{B}} = \overline{\mathfrak{B}}_1.$$

Remark 6. We have

$$\begin{aligned}\mathfrak{B}_k &= \sigma(\{F \subset \mathbb{R}^k : F \text{ closed}\}) = \sigma(\{K \subset \mathbb{R}^k : K \text{ compact}\}) \\ &= \sigma(\{]-\infty, a] : a \in \mathbb{R}^k\}) = \sigma(\{]-\infty, a] : a \in \mathbb{Q}^k\})\end{aligned}$$

and

$$\overline{\mathfrak{B}} = \{B \subset \overline{\mathbb{R}} : B \cap \mathbb{R} \in \mathfrak{B}\}. \tag{4}$$

Moreover,

$$\mathfrak{B}_k \subsetneq \mathfrak{P}(\mathbb{R}^k)$$

since the cardinalities of $\mathfrak{B}_k$ and $\mathbb{R}^k$ coincide, see Billingsley (1979, Exercise 2.21).

Definition 5. For any σ-algebra $\mathfrak{A}$ in Ω and $\widetilde{\Omega} \subset \Omega$

$$\widetilde{\mathfrak{A}} = \{\widetilde{\Omega} \cap A : A \in \mathfrak{A}\}$$

is the *trace-σ-algebra of $\mathfrak{A}$ in $\widetilde{\Omega}$*, sometimes denoted by $\widetilde{\Omega} \cap \mathfrak{A}$.

Remark 7.

(i) $\widetilde{\mathfrak{A}}$ is a σ-algebra.

(ii) $\widetilde{\mathfrak{A}} \not\subset \mathfrak{A}$ in general, but $\widetilde{\Omega} \in \mathfrak{A} \Rightarrow \widetilde{\mathfrak{A}} = \{A \in \mathfrak{A} : A \subset \widetilde{\Omega}\}$.

(iii) $\mathfrak{A} = \sigma(\mathfrak{E}) \Rightarrow \widetilde{\mathfrak{A}} = \sigma(\{\widetilde{\Omega} \cap E : E \in \mathfrak{E}\})$.

(iv) $\mathfrak{B}_k = \mathbb{R}^k \cap \overline{\mathfrak{B}}_k$, see (4) for $k = 1$.

(v) $[a, b[\cap \mathfrak{B}_k = \sigma(\{[a, c[: a \leq c \leq b\})$, see (iii).

2 Measurable Mappings

Definition 1. $(\Omega, \mathfrak{A})$ is called *measurable space* if Ω is a non-empty set and $\mathfrak{A}$ is a σ-algebra in Ω. Elements $A \in \mathfrak{A}$ are called *measurable sets*.

Remark 1. Let $f : \Omega_1 \to \Omega_2$.

(i) $f^{-1}(\mathfrak{A}_2) = \{f^{-1}(A) : A \in \mathfrak{A}_2\}$ is a σ-algebra in Ω_1 for every σ-algebra $\mathfrak{A}_2$ in Ω_2.

(ii) $\{A \subset \Omega_2 : f^{-1}(A) \in \mathfrak{A}_1\}$ is a σ-algebra in Ω_2 for every σ-algebra $\mathfrak{A}_1$ in Ω_1.

In the sequel, $(\Omega_i, \mathfrak{A}_i)$ are measurable spaces for $i = 1, 2, 3$.

Definition 2. $f : \Omega_1 \to \Omega_2$ is $\mathfrak{A}_1$-$\mathfrak{A}_2$-*measurable* if $f^{-1}(\mathfrak{A}_2) \subset \mathfrak{A}_1$.

Example 1. Let $f : \Omega_1 \to \Omega_2$.

(i) Every constant mapping f is $\mathfrak{A}_1$-$\mathfrak{A}_2$-measurable.

(ii) Let $\Omega_2 = \{0, 1\}$ and $\mathfrak{A}_2 = \mathfrak{P}(\Omega_2)$. Then f is $\mathfrak{A}_1$-$\mathfrak{A}_2$-measurable iff $f = 1_A$ with $A \in \mathfrak{A}_1$.

How can we prove measurability of a given mapping?

Theorem 1. If $f : \Omega_1 \to \Omega_2$ is $\mathfrak{A}_1$-$\mathfrak{A}_2$-measurable and $g : \Omega_2 \to \Omega_3$ is $\mathfrak{A}_2$-$\mathfrak{A}_3$-measurable, then $g \circ f : \Omega_1 \to \Omega_3$ is $\mathfrak{A}_1$-$\mathfrak{A}_3$-measurable.

Proof. We have $(g \circ f)^{-1}(\mathfrak{A}_3) = f^{-1}(g^{-1}(\mathfrak{A}_3)) \subset f^{-1}(\mathfrak{A}_2) \subset \mathfrak{A}_1$. □

Lemma 1. For $f : \Omega_1 \to \Omega_2$ and $\mathfrak{E} \subset \mathfrak{P}(\Omega_2)$

$$f^{-1}(\sigma(\mathfrak{E})) = \sigma(f^{-1}(\mathfrak{E})).$$

Proof. By $f^{-1}(\mathfrak{E}) \subset f^{-1}(\sigma(\mathfrak{E}))$ and Remark 1.(i) we get $\sigma(f^{-1}(\mathfrak{E})) \subset f^{-1}(\sigma(\mathfrak{E}))$. Let $\mathfrak{F} = \{A \subset \Omega_2 : f^{-1}(A) \in \sigma(f^{-1}(\mathfrak{E}))\}$. Then $\mathfrak{E} \subset \mathfrak{F}$ and $\mathfrak{F}$ is a σ-algebra, see Remark 1.(ii). Thus we get $\sigma(\mathfrak{E}) \subset \mathfrak{F}$, i.e., $f^{-1}(\sigma(\mathfrak{E})) \subset \sigma(f^{-1}(\mathfrak{E}))$. □

Theorem 2. If $\mathfrak{A}_2 = \sigma(\mathfrak{E})$ with $\mathfrak{E} \subset \mathfrak{P}(\Omega_2)$, then

$$f^{-1}(\mathfrak{E}) \subset \mathfrak{A}_1 \quad \Leftrightarrow \quad f \text{ is } \mathfrak{A}_1\text{-}\mathfrak{A}_2\text{-measurable.}$$

Proof. '$\Rightarrow$': Assume that $f^{-1}(\mathfrak{E}) \subset \mathfrak{A}_1$. By Lemma 1,

$$f^{-1}(\mathfrak{A}_2) = f^{-1}(\sigma(\mathfrak{E})) = \sigma(f^{-1}(\mathfrak{E})) \subset \sigma(\mathfrak{A}_1) = \mathfrak{A}_1.$$

Obviously, '$\Leftarrow$' holds, too. □

Corollary 1. For every pair of topological spaces $(\Omega_1, \mathfrak{G}_1)$ and $(\Omega_2, \mathfrak{G}_2)$ and every mapping $f : \Omega_1 \to \Omega_2$,

$$f \text{ continuous} \quad \Rightarrow \quad f \text{ is } \mathfrak{B}(\Omega_1)\text{-}\mathfrak{B}(\Omega_2)\text{-measurable.}$$

Proof. By assumption,

$$f^{-1}(\mathfrak{G}_2) \subset \mathfrak{G}_1 \subset \sigma(\mathfrak{G}_1) = \mathfrak{B}(\Omega_1).$$

Use Theorem 2. $\square$

Given: measurable spaces $(\Omega_i, \mathfrak{A}_i)$ for $i \in I \neq \emptyset$ and mappings $f_i : \Omega \to \Omega_i$ for $i \in I$ and some non-empty set Ω.

Definition 3. The *σ-algebra generated by $(f_i)_{i\in I}$ (and $(\mathfrak{A}_i)_{i\in I}$)*

$$\sigma(\{f_i : i \in I\}) = \sigma\Big(\bigcup_{i\in I} f_i^{-1}(\mathfrak{A}_i)\Big).$$

Put $\sigma(f) = \sigma(\{f\})$ in the case $|I| = 1$ and $f = f_1$.

Remark 2. $\sigma(\{f_i : i \in I\})$ is the smallest σ-algebra $\mathfrak{A}$ in Ω such that all mappings f_i are $\mathfrak{A}$-$\mathfrak{A}_i$-measurable.

Theorem 3. For every measurable space $(\widetilde{\Omega}, \widetilde{\mathfrak{A}})$ and every mapping $g : \widetilde{\Omega} \to \Omega$,

$$g \text{ is } \widetilde{\mathfrak{A}}\text{-}\sigma(\{f_i : i \in I\})\text{-measurable} \quad \Leftrightarrow \quad \forall\, i \in I : f_i \circ g \text{ is } \widetilde{\mathfrak{A}}\text{-}\mathfrak{A}_i\text{-measurable}.$$

Proof. Use Lemma 1 to obtain

$$g^{-1}(\sigma(\{f_i : i \in I\})) = \sigma\Big(g^{-1}\Big(\bigcup_{i\in I} f_i^{-1}(\mathfrak{A}_i)\Big)\Big) = \sigma\Big(\bigcup_{i\in I} (f_i \circ g)^{-1}(\mathfrak{A}_i)\Big).$$

Therefore

$$g^{-1}\left(\sigma(\{f_i : i \in I\})\right) \subset \widetilde{\mathfrak{A}} \quad \Leftrightarrow \quad \forall\, i \in I : f_i \circ g \text{ is } \widetilde{\mathfrak{A}}\text{-}\mathfrak{A}_i\text{-measurable}.$$

$\square$

Now we turn to the particular case of functions with values in $\mathbb{R}$ or $\overline{\mathbb{R}}$, and we consider the Borel σ-algebra in $\mathbb{R}$ or $\overline{\mathbb{R}}$, respectively. For any measurable space $(\Omega, \mathfrak{A})$ we use the following notation

$$\begin{aligned}
\mathfrak{Z}(\Omega, \mathfrak{A}) &= \{f : \Omega \to \mathbb{R} : f \text{ is } \mathfrak{A}\text{-}\mathfrak{B}\text{-measurable}\}, \\
\mathfrak{Z}_+(\Omega, \mathfrak{A}) &= \{f \in \mathfrak{Z}(\Omega, \mathfrak{A}) : f \geq 0\}, \\
\overline{\mathfrak{Z}}(\Omega, \mathfrak{A}) &= \left\{f : \Omega \to \overline{\mathbb{R}} : f \text{ is } \mathfrak{A}\text{-}\overline{\mathfrak{B}}\text{-measurable}\right\}, \\
\overline{\mathfrak{Z}}_+(\Omega, \mathfrak{A}) &= \left\{f \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A}) : f \geq 0\right\}.
\end{aligned}$$

Every function $f : \Omega \to \mathbb{R}$ may also be considered as a function with values in $\overline{\mathbb{R}}$, and in this case $f \in \mathfrak{Z}(\Omega, \mathfrak{A})$ iff $f \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$.

Corollary 2. For $\prec \in \{\leq, <, \geq, >\}$ and $f : \Omega \to \overline{\mathbb{R}}$,

$$f \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A}) \quad \Leftrightarrow \quad \forall\, a \in \mathbb{R} : \{\omega \in \Omega : f(\omega) \prec a\} \in \mathfrak{A}.$$

Proof. For instance,
$$\{\omega \in \Omega : f(\omega) \leq a\} = f^{-1}([-\infty, a])$$
and $\overline{\mathfrak{B}} = \sigma(\{[-\infty, a] : a \in \mathbb{R}\})$, see Remark 1.6. It remains to apply Theorem 2. □

Theorem 4. For $f, g \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$ and $\prec \in \{\leq, <, \geq, >, =, \neq\}$,
$$\{\omega \in \Omega : f(\omega) \prec g(\omega)\} \in \mathfrak{A}.$$

Proof. For instance, Corollary 2 yields
$$\begin{aligned}\{\omega \in \Omega : f(\omega) < g(\omega)\} &= \bigcup_{q \in \mathbb{Q}} \{\omega \in \Omega : f(\omega) < q < g(\omega)\} \\ &= \bigcup_{q \in \mathbb{Q}} (\{\omega \in \Omega : f(\omega) < q\} \cap \{\omega \in \Omega : g(\omega) > q\}) \in \mathfrak{A}.\end{aligned}$$
□

As is customary, we use the abbreviation
$$\{f \in A\} = \{\omega \in \Omega : f(\omega) \in A\}$$
for any $f : \Omega \to \widetilde{\Omega}$ and $A \subset \widetilde{\Omega}$.

Theorem 5. For every sequence $f_1, f_2, \ldots \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$,

(i) $\inf_{n \in \mathbb{N}} f_n, \ \sup_{n \in \mathbb{N}} f_n \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$,

(ii) $\liminf_{n \to \infty} f_n, \ \limsup_{n \to \infty} f_n \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$,

(iii) if $(f_n)_{n \in \mathbb{N}}$ converges at every point $\omega \in \Omega$, then $\lim_{n \to \infty} f_n \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$.

Proof. For $a \in \mathbb{R}$
$$\left\{\inf_{n \in \mathbb{N}} f_n < a\right\} = \bigcup_{n \in \mathbb{N}} \{f_n < a\}, \quad \left\{\sup_{n \in \mathbb{N}} f_n \leq a\right\} = \bigcap_{n \in \mathbb{N}} \{f_n \leq a\}.$$
Hence, Corollary 2 yields (i). Since
$$\limsup_{n \to \infty} f_n = \inf_{m \in \mathbb{N}} \sup_{n \geq m} f_n, \quad \liminf_{n \to \infty} f_n = \sup_{m \in \mathbb{N}} \inf_{n \geq m} f_n,$$
we obtain (ii) from (i). Finally, (iii) follows from (ii). □

By
$$f^+ = \max(0, f), \quad f^- = \max(0, -f)$$
we denote the positive part and the negative part, respectively, of $f : \Omega \to \overline{\mathbb{R}}$.

Remark 3. For $f \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$ we have $f^+, f^-, |f| \in \overline{\mathfrak{Z}}_+(\Omega, \mathfrak{A})$.

Theorem 6. For $f, g \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$,

$$f \pm g,\ f \cdot g,\ f/g \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A}),$$

provided that these functions are well defined.

Proof. The proof is again based on Corollary 2. For simplicity we only consider the case that f and g are real-valued. Clearly $g \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$ implies $-g \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$, too. Furthermore, for every $a \in \mathbb{R}$,

$$\{f + g < a\} = \bigcup_{q \in \mathbb{Q}} \{f < q\} \cap \{g < a - q\},$$

and therefore $f \pm g \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$. Clearly $f \cdot g \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$ if f is constant. Moreover, $x \mapsto x^2$ defines a $\mathfrak{B}$-$\mathfrak{B}$-measurable function, see Corollary 1, and

$$f \cdot g = 1/4 \cdot \left((f + g)^2 - (f - g)^2\right)$$

We apply Theorem 1 to obtain $f \cdot g \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$ in general. Finally, it is easy to show that $g \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$ implies $1/g \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$. □

Definition 4. $f \in \mathfrak{Z}(\Omega, \mathfrak{A})$ is called *simple function* if $|f(\Omega)| < \infty$. Put

$$\mathfrak{S}(\Omega, \mathfrak{A}) = \{f \in \mathfrak{Z}(\Omega, \mathfrak{A}) : f \text{ simple}\},$$
$$\mathfrak{S}_+(\Omega, \mathfrak{A}) = \{f \in \mathfrak{S}(\Omega, \mathfrak{A}) : f \geq 0\}.$$

Remark 4. $f \in \mathfrak{S}(\Omega, \mathfrak{A})$ iff

$$f = \sum_{i=1}^{n} \alpha_i \cdot 1_{A_i}$$

with $\alpha_1, \dots \alpha_n \in \mathbb{R}$ pairwise different and $A_1, \dots, A_n \in \mathfrak{A}$ pairwise disjoint such that $\bigcup_{i=1}^n A_i = \Omega$.

Theorem 7. For every (bounded) function $f \in \overline{\mathfrak{Z}}_+(\Omega, \mathfrak{A})$ there exists a sequence $f_1, f_2, \dots \in \mathfrak{S}_+(\Omega, \mathfrak{A})$ such that $f_n \uparrow f$ (with uniform convergence).

Proof. Let $n \in N$ and put

$$f_n = \sum_{k=1}^{n \cdot 2^n} \frac{k-1}{2^n} \cdot 1_{A_{n,k}} + n \cdot 1_{B_n}$$

where

$$A_{n,k} = \{(k-1)/(2^n) \leq f < k/(2^n)\}, \quad B_n = \{f \geq n\}.$$

□

Now we consider a mapping $T : \Omega_1 \to \Omega_2$ and a σ-algebra $\mathfrak{A}_2$ in Ω_2. We characterize measurability of functions with respect to $\sigma(T) = T^{-1}(\mathfrak{A}_2)$.

Theorem 8 (Factorization Lemma)**.** For every function $f : \Omega_1 \to \overline{\mathbb{R}}$

$$f \in \overline{\mathfrak{Z}}(\Omega_1, \sigma(T)) \quad \Leftrightarrow \quad \exists\, g \in \overline{\mathfrak{Z}}(\Omega_2, \mathfrak{A}_2) : f = g \circ T.$$

Proof. '$\Leftarrow$' is trivially satisfied. '$\Rightarrow$': First, assume that $f \in \mathfrak{S}_+(\Omega_1, \sigma(T))$, i.e.,

$$f = \sum_{i=1}^n \alpha_i \cdot 1_{A_i}$$

with pairwise disjoint sets $A_1, \dots, A_n \in \sigma(T)$. Take pairwise disjoint sets $B_1, \dots, B_n \in \mathfrak{A}_2$ such that $A_i = T^{-1}(B_i)$ and put

$$g = \sum_{i=1}^n \alpha_i \cdot 1_{B_i}.$$

Clearly $f = g \circ T$ and $g \in \overline{\mathfrak{Z}}(\Omega_2, \mathfrak{A}_2)$.
Now, assume that $f \in \overline{\mathfrak{Z}}_+(\Omega_1, \sigma(T))$. Take a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathfrak{S}_+(\Omega_1, \sigma(T))$ according to Theorem 7. We already know that $f_n = g_n \circ T$ for suitable $g_n \in \overline{\mathfrak{Z}}(\Omega_2, \mathfrak{A}_2)$. Hence

$$f = \sup_n f_n = \sup_n (g_n \circ T) = (\sup_n g_n) \circ T = g \circ T$$

where $g = \sup_n g_n \in \overline{\mathfrak{Z}}(\Omega_2, \mathfrak{A}_2)$.
In the general case, we already know that

$$f^+ = g_1 \circ T, \quad f^- = g_2 \circ T$$

for suitable $g_1, g_2 \in \overline{\mathfrak{Z}}(\Omega_2, \mathfrak{A}_2)$. Put

$$C = \{g_1 = g_2 = \infty\} \in \mathfrak{A}_2,$$

and observe that $T(\Omega_1) \cap C = \emptyset$ since $f = f^+ - f^-$. We conclude that $f = g \circ T$ where

$$g = g_1 \cdot 1_D - g_2 \cdot 1_D \in \overline{\mathfrak{Z}}(\Omega_2, \mathfrak{A}_2)$$

with $D = C^c$. □

Our method of proof for Theorem 8 is sometimes called *algebraic induction.*

3 Product Spaces

Example 1. A stochastic model for coin tossing. For a single trial,

$$\Omega = \{0,1\}, \quad \mathfrak{A} = \mathfrak{P}(\Omega), \quad \forall\, \omega \in \Omega : P(\{\omega\}) = 1/2. \tag{1}$$

For n 'independent' trials, (1) serves as a building-block,

$$\Omega_i = \{0,1\}, \quad \mathfrak{A}_i = \mathfrak{P}(\Omega_i), \quad \forall\, \omega_i \in \Omega_i : P_i(\{\omega_i\}) = 1/2,$$

and we define

$$\Omega = \bigtimes_{i=1}^n \Omega_i, \quad \mathfrak{A} = \mathfrak{P}(\Omega), \quad \forall\, A \in \mathfrak{A} : P(A) = \frac{|A|}{|\Omega|}.$$

Then

$$P(A_1 \times \cdots \times A_n) = P_1(A_1) \cdot \cdots \cdot P_n(A_n)$$

for all $A_i \in \mathfrak{A}_i$.

Question: How to model an infinite sequence of trials? To this end,

$$\Omega = \bigtimes_{i=1}^{\infty} \Omega_i.$$

How to choose a σ-algebra $\mathfrak{A}$ in Ω and a probability measure P on $(\Omega, \mathfrak{A})$? A reasonable requirement is

$$\begin{aligned} &\forall\, n \in \mathbb{N}\ \forall\, A_i \in \mathfrak{A}_i : \\ &\qquad P(A_1 \times \cdots \times A_n \times \Omega_{n+1} \times \Omega_{n+2} \ldots) = P_1(A_1) \cdot \cdots \cdot P_n(A_n). \end{aligned} \tag{2}$$

Unfortunately,

$$\mathfrak{A} = \mathfrak{P}(\Omega)$$

is too large, since there exists no probability measure on $(\Omega, \mathfrak{P}(\Omega))$ such that (2) holds. The latter fact follows from a theorem by Banach and Kuratowski, which relies on the continuum hypothesis, see Dudley (2002, p. 526). On the other hand,

$$\mathfrak{A} = \{A_1 \times \cdots \times A_n \times \Omega_{n+1} \times \Omega_{n+2} \cdots : n \in \mathbb{N},\ A_i \in \mathfrak{A}_i \text{ for } i = 1, \ldots, n\} \tag{3}$$

is not a σ-algebra.

Given: a non-empty set I and measurable spaces $(\Omega_i, \mathfrak{A}_i)$ for $i \in I$. Put

$$Y = \bigcup_{i \in I} \Omega_i$$

and define

$$\bigtimes_{i \in I} \Omega_i = \{\omega \in Y^I : \omega(i) \in \Omega_i \text{ for } i \in I\}.$$

Notation: $\omega = (\omega_i)_{i \in I}$ for $\omega \in \bigtimes_{i \in I} \Omega_i$. Moreover, let

$$\mathfrak{P}_0(I) = \{J \subset I : J \text{ non-empty, finite}\}.$$

The following definition is motivated by (3).

Definition 1.

(i) *Measurable rectangle*

$$A = \bigtimes_{j \in J} A_j \times \bigtimes_{i \in I \setminus J} \Omega_i$$

with $J \in \mathfrak{P}_0(I)$ and $A_j \in \mathfrak{A}_j$ for $j \in J$. Notation: $\mathfrak{R}$ class of measurable rectangles.

(ii) *Product (measurable) space* $(\Omega, \mathfrak{A})$ with *components* $(\Omega_i, \mathfrak{A}_i)$, $i \in I$,

$$\Omega = \bigtimes_{i\in I} \Omega_i, \qquad \mathfrak{A} = \sigma(\mathfrak{R}).$$

Notation: $\mathfrak{A} = \bigotimes_{i\in I} \mathfrak{A}_i$, *product σ-algebra.*

Remark 1. The class $\mathfrak{R}$ is a semi-algebra, but not an algebra in general. See Übung 3.1.

Example 2. Obviously, (2) only makes sense if $\mathfrak{A}$ contains the product σ-algebra $\bigotimes_{i=1}^{n} \mathfrak{A}_i$. We will show that there exists a uniquely determined probability measure P on the product space $\left(\bigtimes_{i=1}^{\infty}\{0,1\}, \bigotimes_{i=1}^{\infty} \mathfrak{P}(\{0,1\})\right)$ that satisfies (2), see Remark **??**.**??**.(ii). The corresponding probability space yields a stochastic model for the simple case of gambling, which was mentioned in the introductory Example I.2.

We study several classes of mappings or subsets that generate the product σ-algebra. Moreover, we characterize measurability of mappings that take values in a product space.

Put $\Omega = \bigtimes_{i\in I} \Omega_i$. For any $\emptyset \neq S \subset I$ let

$$\pi_S^I : \Omega \to \bigtimes_{i\in S} \Omega_i, \quad (\omega_i)_{i\in I} \mapsto (\omega_i)_{i\in S}$$

denote the *projection* of Ω onto $\bigtimes_{i\in S} \Omega_i$ (restriction of mappings ω). In particular, for $i \in I$ the i-th projection is given by $\pi_{\{i\}}^I$. Sometimes we simply write π_S instead of π_S^I and π_i instead of $\pi_{\{i\}}$.

Theorem 1.

(i) $\bigotimes_{i\in I} \mathfrak{A}_i = \sigma(\{\pi_i : i \in I\})$.

(ii) $\forall\, i \in I : \mathfrak{A}_i = \sigma(\mathfrak{E}_i) \quad \Rightarrow \quad \bigotimes_{i\in I} \mathfrak{A}_i = \sigma\left(\bigcup_{i\in I} \pi_i^{-1}(\mathfrak{E}_i)\right)$.

Proof. Ad (i), '$\supset$': We show that every projection $\pi_i : \Omega \to \Omega_i$ is $\left(\bigotimes_{i\in I} \mathfrak{A}_i\right)$-$\mathfrak{A}_i$-measurable. For $A_i \in \mathfrak{A}_i$

$$\pi_i^{-1}(A_i) = A_i \times \bigtimes_{k\in I\setminus\{i\}} \Omega_k \in \mathfrak{R}.$$

Ad (i), '$\subset$': We show that $\mathfrak{R} \subset \sigma(\{\pi_i : i \in I\})$. For $J \in \mathfrak{P}_0(I)$ and $A_j \in \mathfrak{A}_j$ with $j \in J$

$$\bigtimes_{j\in J} A_j \times \bigtimes_{i\in I\setminus J} \Omega_i = \bigcap_{j\in J} \pi_j^{-1}(A_j).$$

Ad (ii): By Lemma 2.1 and (i)

$$\bigotimes_{i\in I} \mathfrak{A}_i = \sigma\Big(\bigcup_{i\in I} \pi_i^{-1}(\mathfrak{A}_i)\Big) = \sigma\Big(\bigcup_{i\in I} \sigma(\pi_i^{-1}(\mathfrak{E}_i))\Big) = \sigma\Big(\bigcup_{i\in I} \pi_i^{-1}(\mathfrak{E}_i)\Big).$$

$\square$

Corollary 1.

(i) For every measurable space $(\widetilde{\Omega}, \widetilde{\mathfrak{A}})$ and every mapping $g : \widetilde{\Omega} \to \Omega$

$$g \text{ is } \widetilde{\mathfrak{A}}\text{-}\bigotimes_{i \in I} \mathfrak{A}_i\text{-measurable} \quad \Leftrightarrow \quad \forall\, i \in I : \pi_i \circ g \text{ is } \widetilde{\mathfrak{A}}\text{-}\mathfrak{A}_i\text{-measurable.}$$

(ii) For every $\emptyset \neq S \subset I$ the projection π_S^I is $\bigotimes_{i\in I} \mathfrak{A}_i$-$\bigotimes_{i \in S} \mathfrak{A}_i$-measurable.

Proof. Ad (i): Follows immediately from Theorem 2.3 and Theorem 1.(i).
Ad (ii): Note that $\pi_{\{i\}}^S \circ \pi_S^I = \pi_i^I$ and use (i). □

Remark 2. From Theorem 1.(i) and Corollary 1 we get

$$\bigotimes_{i \in I} \mathfrak{A}_i = \sigma(\{\pi_S^I : S \in \mathfrak{P}_0(I)\}).$$

The sets

$$\left(\pi_S^I\right)^{-1}(B) = B \times \Big(\bigtimes_{i \in I \setminus S} \Omega_i \Big)$$

with $S \in \mathfrak{P}_0(I)$ and $B \in \bigotimes_{i\in S} \mathfrak{A}_i$ are called *cylinder sets*. Notation: $\mathfrak{C}$ class of cylinder sets. The class $\mathfrak{C}$ is an algebra in Ω, but not a σ-algebra in general. Moreover,

$$\mathfrak{R} \subset \alpha(\mathfrak{R}) \subset \mathfrak{C} \subset \sigma(\mathfrak{R}),$$

where equality does not hold in general.

Every product measurable set is countably determined in the following sense.

Theorem 2. For every $A \in \bigotimes_{i\in I} \mathfrak{A}_i$ there exists a non-empty countable set $S \subset I$ and a set $B \in \bigotimes_{i \in S} \mathfrak{A}_i$ such that

$$A = \left(\pi_S^I\right)^{-1}(B).$$

Proof. Put

$$\widetilde{\mathfrak{A}} = \Big\{ A \in \bigotimes_{i \in I} \mathfrak{A}_i : \exists\, S \subset I \text{ non-empty, countable } \exists\, B \in \bigotimes_{i \in S} \mathfrak{A}_i : A = \left(\pi_S^I\right)^{-1}(B) \Big\}.$$

By definition, $\widetilde{\mathfrak{A}}$ contains every cylinder set and $\widetilde{\mathfrak{A}} \subset \bigotimes_{i \in I} \mathfrak{A}_i$. It remains to show that $\widetilde{\mathfrak{A}}$ is a σ-algebra. See Gänssler, Stute (1977, p. 24) for details. □

Now we study products of Borel-σ-algebras.

Theorem 3.

$$\mathfrak{B}_k = \bigotimes_{i=1}^k \mathfrak{B}, \qquad \overline{\mathfrak{B}}_k = \bigotimes_{i=1}^k \overline{\mathfrak{B}}.$$

Proof. By Remark 1.6,

$$\mathfrak{B}_k = \sigma\Big(\Big\{ \bigtimes_{i=1}^{k} \,]-\infty, a_i] : a_i \in \mathbb{R} \text{ for } i = 1, \dots, k\Big\}\Big) \subset \bigotimes_{i=1}^{k} \mathfrak{B}.$$

On the other hand, $\pi_i : \mathbb{R}^k \to \mathbb{R}$ is continuous, hence it remains to apply Corollary 2.1 and Theorem 1.(i). Analogously, $\overline{\mathfrak{B}}_k = \bigotimes_{i=1}^{k} \overline{\mathfrak{B}}$ follows. □

Remark 3. More generally, consider a non-empty countable set I and a family of topological spaces $(\Omega_i, \mathfrak{G}_i)$ where $i \in I$. Assume that every space $(\Omega_i, \mathfrak{G}_i)$ has a countable basis and consider the product topology $\mathfrak{G}$ on $\Omega = \bigtimes_{i \in I} \Omega_i$. Then

$$\mathfrak{B}(\Omega) = \bigotimes_{i \in I} \mathfrak{B}(\Omega_i),$$

see Gänssler, Stute (1977, Satz 1.3.12).

Remark 4. Consider a measurable space $(\widetilde{\Omega}, \widetilde{\mathfrak{A}})$ and a mapping

$$f = (f_1, \dots, f_k) : \widetilde{\Omega} \to \overline{\mathbb{R}}^k.$$

Then, according to Theorem 3, f is $\widetilde{\mathfrak{A}}$-$\overline{\mathfrak{B}}_k$-measurable iff all functions f_i are $\widetilde{\mathfrak{A}}$-$\overline{\mathfrak{B}}$-measurable.

We briefly discuss the cardinality of σ-algebras. It is known that

$$2 \le |V| \le |\mathbb{R}| \quad \Rightarrow \quad \left|V^{\mathbb{N}}\right| = |\mathbb{R}| \wedge \left|V^{\mathbb{R}}\right| = \left|\{0,1\}^{\mathbb{R}}\right|$$

for every set V, see Hewitt, Stromberg (1965, Exercise 4.34).

Theorem 4. Assume that $\emptyset \in \mathfrak{E} \subset \mathfrak{P}(\Omega)$ and $|\mathfrak{E}| \ge 2$. Then

$$|\sigma(\mathfrak{E})| \le \left|\mathfrak{E}^{\mathbb{N}}\right|.$$

Proof. See Hewitt, Stromberg (1965, Theorem 10.13). □

Example 3. Let $I = \mathbb{N}$, $\Omega_i = \{0,1\}$, and $\mathfrak{A}_i = \mathfrak{P}(\Omega_i)$, as in Example 1. For the corresponding product space $(\Omega, \mathfrak{A})$ we have $\Omega = \{0,1\}^{\mathbb{N}}$ and

$$|\mathfrak{A}| = |\Omega| = |\mathbb{R}|.$$

Proof: Note that $\{\omega\} \in \mathfrak{A}$ for every $\omega \in \Omega$. Hence $|\mathfrak{A}| \ge |\Omega|$. Conversely, use Theorem 1.(ii) with $\mathfrak{E}_i = \{\{1\}\}$ and Theorem 4 to conclude that $|\mathfrak{A}| \le |\mathbb{N}^{\mathbb{N}}| = |\mathbb{R}|$.
We add that $|\mathfrak{P}(\Omega)| = |\{0,1\}^{\mathbb{R}}| > |\Omega|$.

Example 4. Let $I = \mathbb{N}$, $\Omega_i = \mathbb{R}$, and $\mathfrak{A}_i = \mathfrak{B}$. For the corresponding product space $(\Omega, \mathfrak{A})$ we have $\Omega = \mathbb{R}^{\mathbb{N}}$ and

$$|\mathfrak{A}| = |\Omega| = |\mathbb{R}|.$$

Proof: As in the previous example, with $\mathfrak{E}_i = \{]-\infty, a] : a \in \mathbb{Q}\}$.
Again we have $|\mathfrak{P}(\Omega)| = |\{0,1\}^{\mathbb{R}}| > |\Omega|$.
The sets $\{(x_n)_{n\in\mathbb{N}} : (x_n)_{n\in\mathbb{N}} \text{ converges}\}$ and $\{(x_n)_{n\in\mathbb{N}} : (x_n)_{n\in\mathbb{N}} \text{ is bounded}\}$ are elements of $\mathfrak{A}$, but they are not cylinder sets.

Example 5. Let $I = \mathbb{R}_+$, $\Omega_i = \mathbb{R}$, and $\mathfrak{A}_i = \mathfrak{B}$. For the corresponding product space $(\Omega, \mathfrak{A})$ we have $\Omega = \mathbb{R}^{\mathbb{R}_+}$ and

$$|\mathfrak{A}| = |\mathbb{R}| < |\Omega|.$$

Proof: Clearly $|\mathbb{R}| \leq |\mathfrak{A}|$ and $|\mathbb{R}| < |\Omega|$. On the other hand, Theorem 2 shows that $\mathfrak{A} = \sigma(\mathfrak{E})$ for some set $\mathfrak{E}$ with $|\mathfrak{E}| = |\mathbb{R}|$. Hence $|\mathfrak{A}| \leq |\mathbb{R}|$ by Theorem 4.

The space $\mathbb{R}^{\mathbb{R}_+}$ already appeared in the introductory Example I.3. The product σ-algebra $\mathfrak{A} = \bigotimes_{i \in \mathbb{R}_+} \mathfrak{B}$ is a proper choice on this space. On the subspace $C(\mathbb{R}_+) \subset \mathbb{R}^{\mathbb{R}_+}$ we can take the trace-σ-algebra. It is important to note, however, that

$$C(\mathbb{R}_+) \notin \mathfrak{A},$$

see Übung 3.2. It turns out that the Borel σ-algebra $\mathfrak{B}(C(\mathbb{R}_+))$ that is generated by the topology of uniform convergence on compact intervals coincides with the trace-σ-algebra of $\mathfrak{A}$ in $C(\mathbb{R}_+)$, see Bauer (1996, Theorem 38.6).

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Index