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8. Aufgabenblatt zur Vorlesung „Probability Theory“

1. Let f_n and f denote probability densities w.r.t. the Lebesgue measure λ_1.

a) Suppose that f_n converges to f λ_1-almost everywhere. Show that $f_n \cdot \lambda_1$ converges weakly to $f \cdot \lambda_1$.

b) Provide an example, where $f_n \cdot \lambda_1$ converges weakly to $f \cdot \lambda_1$ but f_n converges to f only on a set of Lebesgue measure zero.

2. Let

$$Q_n = N(\mu_n, \sigma_n^2)$$

where $\mu_n \in \mathbb{R}$ and $\sigma_n \geq 0$.

a) Show that $(Q_n)_{n\in\mathbb{N}}$ converges weakly iff $(\mu_n)_{n\in\mathbb{N}}$ and $(\sigma_n)_{n\in\mathbb{N}}$ converge. Determine the weak limit in case of convergence.

b) Characterize tightness of the set $\{Q_n : n \in \mathbb{N}\}$.

3. Consider random variables $X_n,\ X,\ Y_n,\ Y$ on a common probability space. Prove or disprove

a) $X_n \xrightarrow{\mathrm{d}} X \quad \wedge \quad Y_n \xrightarrow{\mathrm{d}} Y \quad \Rightarrow \quad X_n + Y_n \xrightarrow{\mathrm{d}} X + Y.$

b) $X, X_n \in \mathfrak{L}^1 \quad \wedge \quad X_n \xrightarrow{\mathrm{d}} X \quad \wedge \quad (E(X_n))_{n\in\mathbb{N}}$ converges $\quad \Rightarrow \quad \lim_{n\to\infty} E(X_n) = E(X).$

c) $(X_n)_{n\in\mathbb{N}}$ u.i. $\quad \wedge \quad X \in \mathfrak{L}^1 \quad \wedge \quad X_n \xrightarrow{\mathrm{d}} X \quad \Rightarrow \quad X_n \xrightarrow{\mathfrak{L}^1} X.$

4. Consider the set

$$\mathfrak{P} = \left\{ \sum_{k=1}^{n} \lambda_k \cdot \varepsilon_{x_k} : n \in \mathbb{N},\ \lambda_k > 0,\ \sum_{k=1}^{n} \lambda_k = 1,\ x_k \in \mathbb{R} \right\}$$

of probability measures on $(\mathbb{R}, \mathfrak{B})$. Prove that $\mathfrak{P}$ is dense in the set of all probability measures on $(\mathbb{R}, \mathfrak{B})$ w.r.t. weak convergence, i.e., for every probability measure μ on $(\mathbb{R}, \mathfrak{B})$ a suitable sequence in $\mathfrak{P}$ converges weakly to μ.