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6. Aufgabenblatt zur Vorlesung „Probability Theory“

1. Let λ denote the Lebesgue measure on $(\mathbb{R}, \mathfrak{B})$.

a) Put

$$\begin{aligned} f_1(x) &= \frac{1}{\sqrt{2\pi}} \cdot \exp(-x^2/2), \\ f_2(x) &= \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp(-(x-m)^2/(2\sigma^2)), \\ f_3(x) &= 1_{\mathbb{R}_+}(x) \cdot \exp(-x), \end{aligned}$$

where $\sigma > 0$ and $m \in \mathbb{R}$. For which choice of $i \neq j$ do we have $f_i \cdot \lambda \ll f_j \cdot \lambda$? Compute the corresponding densities.

b) Let μ denote the counting measure on $\mathfrak{B}$. Show that $\lambda \neq f \cdot \mu$ for every $f \in \overline{\mathfrak{Z}}_+(\mathbb{R}, \mathfrak{B})$.

2. Beweisen Sie den Satz von Radon-Nikodym im Falle endlicher Maße ν und μ mit $\nu \ll \mu$ unter Verwendung der in der Vorlesung bewiesenen Variante des Satzes.

3. a) Let $\Omega_1 = \Omega_2 = \mathbb{N}$, $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{P}(\mathbb{N})$, $\mu_1 = \mu_2$ be counting measure, and consider the function

$$f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}, \quad (m,n) \mapsto \begin{cases} 1 - 2^{-n} & \text{if } m = n \\ 2^{-n} - 1 & \text{if } m = n+1 \\ 0 & \text{otherwise.} \end{cases}$$

Show that the iterated integrals

$$\int_{\Omega_1} \int_{\Omega_2} f(m,n)\, \mu_2(dn)\, \mu_1(dm), \qquad \int_{\Omega_2} \int_{\Omega_1} f(m,n)\, \mu_1(dm)\, \mu_2(dn)$$

exist but are unequal. Why doesn't this contradict Fubini's theorem?

b) Sei $\Omega_1 = \Omega_2 = \mathbb{R}$, $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{B}$, μ das Zählmaß, und f die Indikatorfunktion der Diagonale in $\mathbb{R}^2$. Zeigen Sie, dass $f \in \mathcal{Z}_+(\mathbb{R}^2, \mathcal{B}_2)$, aber

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(x,y)\, \mu(dx)\, \lambda(dy) \neq \int_{\mathbb{R}} \int_{\mathbb{R}} f(x,y)\, \lambda(dy)\, \mu(dx).$$

4. Consider a measure space $(\Omega, \mathfrak{A}, \mu)$ with a σ-finite measure μ and a function $f \in \overline{\mathfrak{Z}}_+(\Omega, \mathfrak{A})$. Show that

$$\int f \, d\mu = \int_{]0,\infty[} \mu(\{f > x\})\, \lambda_1(dx).$$