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5. Aufgabenblatt zur Vorlesung „Probability Theory“

1. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Show that if $f \in \mathcal{L}^1(\Omega, \mathcal{A}, \mu)$, then for each $\varepsilon > 0$ there is a $\delta > 0$ such that $\mu(A) < \delta$ implies $\int_A |f| \, d\mu < \varepsilon$.

2. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $f \in \mathcal{L}^1(\Omega, \mathcal{A}, \mu)$. Show that for $\varepsilon > 0$ there exists a μ-integrable function $g = \sum_{i=1}^n \alpha_i \cdot 1_{A_i} \in \mathcal{S}(\Omega, \mathcal{A})$ such that

$$\int |f - g| \, d\mu < \varepsilon.$$

Moreover, if μ is finite and $\mathcal{A}$ is generated by an algebra $\mathcal{A}_0 \subset \mathcal{P}(\Omega)$ then g can be taken with $A_1, \dots, A_n \in \mathcal{A}_0$.

In the sequel, the underlying measure space is denoted by $(\Omega, \mathfrak{A}, \mu)$.

3. Let $f_n, f \in \mathfrak{L}^\infty$. Prove or disprove $f_n \xrightarrow{\mathfrak{L}^\infty} f \Rightarrow f_n \xrightarrow{\mu\text{-a.e.}} f$ and $f_n \xrightarrow{\mu\text{-a.e.}} f \Rightarrow f_n \xrightarrow{\mathfrak{L}^\infty} f$.

4. Let $p, q \in [1, \infty]$ such that $1/p + 1/q = 1$. Moreover, let $f_n, f \in \mathfrak{L}^p$ and $g_n, g \in \mathfrak{L}^q$ such that $f_n \xrightarrow{\mathfrak{L}^p} f$ and $g_n \xrightarrow{\mathfrak{L}^q} g$.

a) Show that $f_n \cdot g_n \xrightarrow{\mathfrak{L}^1} f \cdot g$.

b) Let $A \in \mathfrak{A}$ and assume that $\mu(A) < \infty$ or $p = 1$. Show that

$$\lim_{n \to \infty} \int_A f_n \, d\mu = \int_A f \, d\mu.$$