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4. Aufgabenblatt zur Vorlesung „Probability Theory“

1. Consider the stochastic model $(\Omega, \mathcal{A}, P)$ for coin tossing with an infinite sequence of trials, see Remark II.4.3.(ii).

a) Show that $\{\omega\} \in \mathcal{A}$ and $P(\{\omega\}) = 0$ for every $\omega \in \Omega$.

b) Let $S_1, \ldots, S_n$ denote pairwise disjoint sets in $\mathcal{P}_0(\mathbb{N})$ and let $A_j \in \mathcal{P}(\{0,1\}^{S_j})$. Show that

$$P\Big(\bigcap_{j=1}^{n} \pi_{S_j}^{-1}(A_j)\Big) = \prod_{j=1}^{n} P(\pi_{S_j}^{-1}(A_j)).$$

c) Let $S_j = \{1\}$ and $S_j = \{\binom{j}{2} + 1, \ldots, \binom{j+1}{2}\}$ for $j \geq 2$. Show that

$$0 < P(\{\omega \in \Omega : \forall\, j \in \mathbb{N}\ \exists\, i \in S_j :\ w_i = 0\}) < 1.$$

d) Is $(\Omega, \mathcal{A}, P)$ a complete measure space?

2. Let $(\Omega, \mathcal{A}^\mu, \widetilde{\mu})$ be the completion of the measure space $(\Omega, \mathcal{A}, \mu)$. For $A \in \mathcal{P}(\Omega)$ define

$$\mu_\dagger(A) = \sup\{\mu(B) :\ B \subset A,\ B \in \mathcal{A}\}, \quad \mu^\dagger(A) = \inf\{\mu(C) :\ A \subset C,\ C \in \mathcal{A}\},$$

and put

$$\mathcal{A}^* = \{A \in \mathcal{P}(\Omega) :\ \mu_\dagger(A) = \mu^\dagger(A)\}.$$

a) Show that

$$\{A \in \mathcal{A}^* :\ \mu_\dagger(A) < \infty\} \subset \mathcal{A}^\mu \subset \mathcal{A}^*$$

and

$$\widetilde{\mu}(A) = \mu_\dagger(A) = \mu^\dagger(A) \quad \text{for all } A \in \mathcal{A}^\mu.$$

b) Show that $\mathcal{A}^\mu = \mathcal{A}^*$ does not hold in general.

3. Consider the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$, where μ is the counting measure.

a) Let $f : \mathbb{N} \to \mathbb{R}$. Show that

$$\sum_{n\geq 1} f(n) \text{ converges absolutely} \quad \text{iff} \quad f \text{ is } \mu\text{-integrable.}$$

In this case

$$\int f\, d\mu = \sum_{n\geq 1} f(n).$$

b) Formulate the ‘dominated convergence theorem’ in this particular situation.

4. Sei $(\Omega, \mathcal{A})$ ein messbarer Raum und $\mu_n,\ n \geq 1$, Maße auf $\mathcal{A}$. Zeigen Sie:

a) Die Mengenfunktion $\sum_{n\geq 1} \mu_n :\ \mathcal{A} \to [0, \infty]$ mit

$$\Big(\sum_{n\geq 1} \mu_n\Big)(A) = \sum_{n\geq 1} \mu_n(A), \qquad A \in \mathcal{A},$$

ist ein Maß auf $\mathcal{A}$.

b) Sei $f \in \overline{\mathcal{Z}}(\Omega, \mathcal{A})$. Ist f $\sum_{n\geq 1} \mu_n$-quasi-integrierbar, so ist f μ_n-quasi-integrierbar für alle $n \in \mathbb{N}$ und es gilt

$$\int f \; d\Big(\sum_{n\geq 1} \mu_n\Big) = \sum_{n\geq 1} \int f \; d\mu_n.$$

Sehen Sie einen Zusammenhang mit der Aufgabe 3?