TU Darmstadt
Fachbereich Mathematik
Klaus Ritter

WS 2008/09
17.12.08

10. Aufgabenblatt zur Vorlesung „Probability Theory“

1. Verify the facts that are stated in Remark IV.1.2.

2. Consider a sequence $(X_n)_{n\in\mathbb{N}}$ of random variables according to Example IV.1.3 with $p = 1/2$.

a) Show that

$$P(\{\liminf_{n\to\infty} S_n/\sqrt{n} \le -x\} \cap \{\limsup_{n\to\infty} S_n/\sqrt{n} \ge x\}) = 1$$

for every $x > 0$.

b) Conclude that $P(\limsup_{n\to\infty}\{S_n = 0\}) = 1$.

3. Consider

- a set $D \in \mathfrak{B}_d$ with $0 < \lambda_d(D) < \infty$,
- a square-integrable function $f : D \to \mathbb{R}$,
- a sequence $(U_n)_{n\in\mathbb{N}}$ that is i.i.d. with U_1 being uniformly distributed on D.

Put

$$a = \int_D f(x)\,dx, \qquad M_n = \frac{\lambda_d(D)}{n} \cdot \sum_{k=1}^{n} f \circ U_k, \qquad \Delta_n = a - M_n.$$

a) Show that

$$E(\Delta_n)^2 \le \frac{\lambda_d(D)}{n} \cdot \int_D f^2(x)\,dx.$$

b) Suppose you only know a constant $c > 0$ such that $|f(x)| \le c$ for every $x \in D$. Determine an integer n_0 such that

$$P\left(\left\{|\Delta_n| \ge 10^{-2}\right\}\right) \le 10^{-4}$$

for every $n \ge n_0$.

c) Let $\alpha \in]0, 1/2[$. Show that

$$\lim_{n\to\infty} n^\alpha \cdot \Delta_n = 0$$

with probability one.

4. Perform numerical experiments for Monte Carlo integration in the case $D = [0,1]^d$. Use uniformly distributed random numbers from $[0,1]$ that are available on your computer. Take into account the results from Exercise 3.

Consider, in particular, test functions

$$f(x) = \exp\Bigl(-\sum_{j=1}^{d} c_j \cdot |x_j - w_j|\Bigr)$$

with constants $c_j > 0$ and $0 < w_j < 1$ (which are easily integrated exactly.)