IV. General Lie groups

In this chapter we introduce the concept of a general Lie group: a smooth manifold which carries a group structure for which multiplication and inversion are smooth maps. In the second section we shall provide some useful constructions dealing with the creation of Lie group structures from, a priori, weaker data.

IV.1. Differentiable manifolds and Lie groups

We have seen in Chapter III that linear Lie groups, i.e., closed subgroups of $\mathrm{GL}_n(\mathbb{K})$, are submanifolds of the vector space $M_n(\mathbb{K})$. On the other hand subgroups such as the dense wind of the 2-dimensional torus $\mathbb{T}^2$ which are not closed are not submanifolds, hence do not have any obvious differentiable structure.

In this section we shall turn to the relation between Lie groups and differentiable manifolds in the abstract sense. Contrary to submanifolds of some vector space, a differentiable manifold is described without specifying any surrounding space in which it is embedded. In spite of the fact that one can show that each differentiable manifold can be realized as a closed submanifold of some $\mathbb{R}^n$ (Whitney's Embedding Theorem), these embeddings are not canonical, and it is therefore much more natural to think of differentiable manifolds as spaces per se, for which no embedding is specified. The concept of a differentiable manifold permits us to define a Lie group as a differentiable manifold for which the group operations are smooth maps. We shall verify below that this approach is compatible with the concept discussed in the preceding chapter.

Definition IV.1.1. Let M be a Hausdorff space. A *k-dimensional smooth atlas on M* is a family $\mathcal{A}$ of pairs $(U_\alpha, \varphi_\alpha)_{\alpha\in I}$ with the following properties:

(M1) Each U_α is an open subset of M and $M = \bigcup_{\alpha\in I} U_\alpha$.

(M2) Each φ_α is a homeomorphism $\varphi_\alpha\colon U_\alpha \to \varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^k$ onto an open subset.

(M3) For each pair $\alpha, \beta \in I$, the transition map

$$\varphi_\alpha \circ \varphi_\beta^{-1}\colon \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$

is a smooth ($=C^\infty$) map between open subsets of $\mathbb{R}^k$.

The pairs $(U_\alpha, \varphi_\alpha)$ are called *charts* of M and the condition (M3) means that two charts of the atlas are *compatible*. We call an atlas $\mathcal{A}$ on M *maximal* if it contains all charts (ψ, V) compatible with all charts of $\mathcal{A}$. A maximal atlas on

M is called a *differentiable structure*, and the pair $(M, \mathcal{A})$ is called a *differentiable manifold* or a *smooth manifold*. The number k is called the *dimension of* M. In the following we simply write M for $(M, \mathcal{A})$ and call the charts in the atlas $\mathcal{A}$ simply the charts of M. ■

Remark IV.1.2. (a) A given topological space M might carry different differentiable structures. Examples are the exotic differentiable structures on $\mathbb{R}^4$ (the only $\mathbb{R}^n$ carrying exotic differentiable structures) and the 7-sphere $\mathbb{S}^7$.

(b) Each atlas $\mathcal{A}$ on a Hausdorff space M is contained in a maximal atlas. One simply has to enlarge $\mathcal{A}$ by all the charts compatible with all charts in $\mathcal{A}$. It is easy to verify that one thus obtains an atlas. To specify the structure of a differentiable manifold on M, it therefore suffices to specify one atlas. ■

Examples IV.1.3. (a) Each space $M := \mathbb{R}^n$ is a differentiable manifold, where the differentiable structure is given by the atlas $(\mathbb{R}^n, \varphi)$, where $\varphi = \mathrm{id}_{\mathbb{R}^n}$. If V is a finite-dimensional real vector space, then we obtain a canonical manifold structure on V by specifying a linear isomorphism $\varphi: V \to \mathbb{R}^n$ and using the chart (V, φ).

(b) Each open subset U of a finite-dimensional real vector space $V \cong \mathbb{R}^n$ is a manifold. A chart is obtained by (U, φ), where φ is the restriction to U of a linear isomorphism $V \to \mathbb{R}^n$.

(c) If $M \subseteq \mathbb{R}^n$ is a k-dimensional submanifold, then M is a Hausdorff space with respect to the topology inherited from $\mathbb{R}^n$. To obtain an atlas on M, we consider for each $x \in M$ an open neighborhood V_x of x in $\mathbb{R}^n$ and a diffeomorphism $\psi: V_x \to W$ onto an open neighborhood W of 0 in $\mathbb{R}^n$ such that

$$\psi(V_x \cap M) = W \cap \mathbb{R}^k.$$

We define $U_x := M \cap V_x$ and $\varphi_x := \psi\,|_{U_x \cap M}: U_x \cap M \to \mathbb{R}^k$. Then $\mathcal{A} := (U_x, \varphi_x)_{x \in M}$ is an atlas of M. This requires some verification (Exercise).

(d) An important example of a smooth manifold is the *sphere*

$$\mathbb{S}^n := \{x \in \mathbb{R}^{n+1}: \|x\|_2 = 1\}.$$

For each $i \in \{1, \ldots, n+1\}$, we consider the open subsets

$$U_{i,\pm} := \{x \in \mathbb{S}^n: \pm x_i > 0\}.$$

It is clear that the open sets $U_{i,\pm}$ cover $\mathbb{S}^n$. On each $U_{i,\pm}$ we define a chart

$$\varphi_i: U_{i,\pm} \to \mathbb{R}^n, \quad (x_0, \ldots, x_n) \mapsto (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

Then each φ_i is a homeomorphism of $U_{i,\pm}$ onto the open unit ball B in $\mathbb{R}^n$, and

$$\varphi_i^{-1}: B \to U_{i,\pm}, \quad y \mapsto \Big(y_0, \ldots, y_{i-1}, \pm\big(1 - \sum_{j \neq i} y_j^2\big)^{\frac{1}{2}}, y_{i+1}, \ldots, y_{n+1}\Big).$$

It is easy to verify that the charts $(\varphi_i, U_{i,\pm})$ form a smooth atlas on $\mathbb{S}^n$. ■

Definition IV.1.4. (a) Let M and N be differentiable manifolds. We call a continuous map $f\colon M \to N$ *smooth in* $p \in M$ if for each chart (U, φ) of N and each chart (V, ψ) of M with $p \in V \cap f^{-1}(U)$ the map

$$\varphi \circ f \circ \psi^{-1}\colon \psi(f^{-1}(U) \cap V) \to \varphi(U), \quad \psi(x) \mapsto \varphi(f(x)) \tag{1.1}$$

between open subsets of a vector space is smooth in a neighborhood of $\psi(p)$. Note that the assumption that f is continuous implies that $f^{-1}(U)$ is open in M, so that the set $\psi(f^{-1}(U) \cap V)$ is open. We call a continuous map $f\colon M \to N$ *smooth* if it is smooth in each point of M.

(b) A smooth map $f\colon M \to N$ is called a *differentiable isomorphism* or a *diffeomorphism* if there exists a smooth map $g\colon N \to M$ with $g \circ f = \mathrm{id}_M$ and $f \circ g = \mathrm{id}_N$. ■

Remark IV.1.5. (a) To verify the smoothness of a continuous map $f\colon M \to N$ between differentiable manifolds in $x \in M$ one does not have to consider all charts of M and N. If suffices that for one chart (V, ψ) of M with $x \in V$ and one chart (U, φ) of N with $f(x) \in U$ the map in (1.1) is smooth (Exercise).

(b) The map $f\colon \mathbb{R} \to \mathbb{R}, x \mapsto x^3$ is smooth and invertible, but it is not a diffeomorphism because f^{-1} is not differentiable in 0. ■

Definition IV.1.6. (Product manifolds) Let M and N be differentiable manifolds with atlases $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ and $(V_\beta, \psi_\beta)_{\beta \in B}$. Then the topological product $M \times N$ is a Hausdorff space and we obtain the structure of a smooth manifold on $M \times N$ by the atlas $(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta)_{(\alpha,\beta) \in A \times B}$, where

$$(\varphi_\alpha \times \psi_\beta)\colon U_\alpha \times V_\beta \to \varphi_\alpha(U_\alpha) \times \psi_\beta(V_\beta), \quad (x, y) \mapsto (\varphi_\alpha(x), \psi_\beta(y))$$

(cf. Exercise IV.1.5). ■

Remark IV.1.7. The projection maps $p_M\colon M \times N \to M$ and $p_N\colon M \times N \to N$ are smooth maps. ■

Geometric definition of a Lie group

Definition IV.1.8. A *Lie group* G is a smooth manifold for which the group operations

$$m_G\colon G \times G \to G, \quad (x, y) \mapsto xy \quad \text{and} \quad \eta_G\colon G \to G, \quad x \mapsto x^{-1}$$

are smooth maps. ■

Remark IV.1.9. (a) It is easy to see that the smoothness requirements in the definition of a Lie group are equivalent to the requirement that the map

$$G \times G \to G, \quad (x, y) \mapsto xy^{-1}$$

is smooth. ■

Example IV.1.10. Let $(E,+)$ be the additive group of a finite-dimensional vector space E and $\varphi\colon E \to \mathbb{R}^n$ a linear isomorphism. Then the chart (E,φ) defines on E the structure of a differentiable manifold (Examples IV.1.3), and it is obvious that the group operations on E are smooth maps. This implies that $(E,+)$ is a Lie group. ∎

Lemma IV.1.11. *Let G be a Lie group and $g \in G$. Then the following maps are diffeomorphisms of G:*

(1) $\lambda_g\colon G \to G, x \mapsto gx$ *(left translations).*

(2) $\rho_g\colon G \to G, x \mapsto xg$ *(right translations).*

(3) $c_g\colon G \to G, x \mapsto gxg^{-1}$ *(conjugations).*

Proof. The smoothness of all these maps follows from the smoothness of the group operations. That they are diffeomorphisms is a consequence of their bijectivity and $\lambda_g^{-1} = \lambda_{g^{-1}}$, $\rho_g^{-1} = \rho_{g^{-1}}$ and $c_g^{-1} = c_{g^{-1}}$. ∎

Without proof we mention the following theorem which provides for general finite-dimensional Lie groups the same structure that we have developed in Chapter III for linear Lie groups.

Theorem IV.1.12. *For every Lie group G there exists a Lie algebra $\mathbf{L}(G)$ and a smooth function*

$$\exp_G\colon \mathbf{L}(G) \to G$$

with the following properties. There exists an open 0-neighborhood $V \subseteq \mathbf{L}(G)$ on which the Dynkin series converges and $\exp_G|_V$ is a diffeomorphism onto an open subset of G with

$$\exp_G(x * y) = \exp_G(x)\exp_G(y) \quad \text{for} \quad x, y \in V. \qquad ∎$$

There are many ways to prove the preceding theorem. First one has to define the Lie algebra $\mathbf{L}(G)$, which is usually done by taking the space of left invariant vector fields on G (a concept that we have not developed in this course). Then one shows that all these vector fields are complete, i.e., generate global flows and then the exponential function is obtained as the time-1 evaluation of the trajectories starting in $\mathbf{1}$. The calculation of the differential of the exponential function and the BCH formalism now essentially follow the same lines as for linear Lie groups.

The preceding theorem has the same consequences for general Lie groups that we know already for linear Lie groups.

Corollary IV.1.13. *If G is a Lie group and $x, y \in \mathbf{L}(G)$, then we have the Trotter Product Formula*

$$\lim_{k\to\infty}\Big(\exp_G\big(\frac{x}{k}\big)\exp_G\big(\frac{y}{k}\big)\Big)^k = \exp_G(x+y)$$

and the Commutator Formula

$$\lim_{k\to\infty}\Big(\exp_G\frac{x}{k}\exp_G\frac{y}{k}\exp_G-\frac{x}{k}\exp_G-\frac{y}{k}\Big)^{k^2} = \exp_G([x,y]). \qquad ∎$$

Theorem IV.1.14. (Closed Subgroup Theorem) *Each closed subgroup H of a Lie group G carries a Lie group structure with*

$$\mathbf{L}(H) \cong \{x \in \mathbf{L}(G) \colon \exp_G(\mathbb{R}x) \subseteq H\}$$

for which

$$\exp_H := \exp_G|_{\mathbf{L}(H)} \colon \mathbf{L}(H) \to H$$

is the exponential function of H. In particular, the inclusion $H \hookrightarrow G$ is a topological embedding. ■

Lemma IV.1.15. *Let G and H be Lie groups and $\varphi \colon G \to H$ a group homomorphism. Then the following are equivalent:*

(1) *φ is smooth.*

(2) *φ is smooth in some open $\mathbf{1}$-neighborhood of G.*

Proof. We only have to show that (2) implies (1). So let U be an open $\mathbf{1}$-neighborhood of G such that $\varphi|_U$ is smooth. Since each left translation λ_g is a diffeomorphism, $\lambda_g(U) = gU$ is an open neighborhood of g, and we have

$$\varphi(gx) = \varphi(g)\varphi(x), \quad \text{i.e.,} \quad \varphi \circ \lambda_g = \lambda_{\varphi(g)} \circ \varphi.$$

Hence the smoothness of φ on U implies the smoothness of φ on gU, and therefore that φ is smooth. ■

Theorem IV.1.16. (Automatic Smoothness Theorem) *Each continuous homomorphism $\varphi \colon G \to H$ of Lie groups is smooth and there exists a unique morphism*

$$\mathbf{L}(\varphi) \colon \mathbf{L}(G) \to \mathbf{L}(H)$$

of Lie algebras with

$$\exp_H \circ \mathbf{L}(\varphi) = \varphi \circ \exp_G. \tag{1.2}$$

We further have:

(1) *The restriction of φ to the identity component G_0 is uniquely determined by $\mathbf{L}(\varphi)$.*

(2) *$\mathbf{L}(\ker \varphi) = \ker \mathbf{L}(\varphi)$.*

(3) *If φ is an open map, then $\mathbf{L}(\varphi)$ is surjective.*

Proof. (Sketch) Using Theorem IV.1.12 and Corollary IV.1.13, one first observes that the map

$$\mathbf{L}(G) \to \operatorname{Hom}(\mathbb{R}, G), \quad x \mapsto \gamma_x, \quad \gamma_x(t) := \exp_G(tx)$$

is a bijection, where $\operatorname{Hom}(\mathbb{R}, G)$ denotes the set of all continuous one-parameter groups of G (cf. Theorem II.2.6). Now we argue as in the proof of Theorem III.1.8 to obtain $\mathbf{L}(\varphi)$ satisfying (1.2), and the Trotter Product Formula and the Commutator Formula imply that $\mathbf{L}(\varphi)$ is a homomorphism of Lie algebras.

Since $\exp_G$ and $\exp_H$ are local diffeomorphisms in 0, the smoothness of the linear map $\mathbf{L}(\varphi)$ now implies that φ is smooth in an identity neighborhood of G, hence smooth (Lemma IV.1.15).

(1) In view of Exercise IV.1.2(3), the identity component G_0 of G is generated by the image of the exponential function of G, and since the values of φ on elements of the form $\exp_G(x)$ is determined by $\mathbf{L}(\varphi)$ via (1.2), $\mathbf{L}(\varphi)$ determined $\varphi|_{G_0}$.

(2) The condition $x \in \ker \mathbf{L}(\varphi)$ is equivalent to

$$\{\mathbf{1}\} = \exp(\mathbb{R}\,\mathbf{L}(\varphi)x) = \varphi(\exp_G(\mathbb{R}x)).$$

(3) Suppose that φ is an open map. Since $\exp_G$ and $\exp_H$ are local diffeomorphisms, there exists some 0-neighborhood in $\mathbf{L}(G)$ on which $\mathbf{L}(\varphi)$ is an open map, hence that $\mathbf{L}(\varphi)$ is surjective. ■

Exercises for Section IV.1.

Exercise IV.1.1. Every open subset of a smooth manifold carries a canonical structure of a smooth manifold. ■

Exercise IV.1.2. Let G be a topological group and $H \leq G$ a subgroup. Show that:

(1) If H is a neighborhood of $\mathbf{1}$, then H is an open subgroup. Hint: $H = H \cdot H = \bigcup_{h \in H} hH$.

(2) Show that each open subgroup H of a topological group G is closed. Hint: The complement of H is a union of open cosets.

(3) If G is a Lie group, then the connected component of the identity is an open subgroup. It coincides with $\langle \exp \mathbf{L}(G) \rangle = \bigcup_{n \in \mathbb{N}} (\exp \mathbf{L}(G))^n$. ■

Exercise IV.1.3. Suppose that M and N are smooth manifolds and that $M \times N$ is the product manifold. Verify the universal property of the product structure: For a smooth manifold L, a map $f\colon L \to M \times N$ is smooth if and only if the two maps $p_M \circ f\colon L \to M$ and $p_N \circ f\colon L \to N$ are smooth. Here $p_M\colon M \times N \to M$ and $p_N\colon M \times N \to N$ denote the projections. ■

Exercise IV.1.4. In the definition of a smooth manifold we start with a Hausdorff space M with an atlas. An alternative approach would be to start with a set M (no topology) and a family $(U_\alpha, \varphi_\alpha)_{\alpha \in I}$ of maps $\varphi_\alpha\colon U_\alpha \to \mathbb{R}^k$ with the following properties:

(M1') $M = \bigcup_{\alpha \in I} U_\alpha$.

(M2') Each φ_α is a bijection $\varphi_\alpha\colon U_\alpha \to \varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^k$ onto an open subset.

(M3') For each pair $\alpha, \beta \in I$, the transition map

$$\varphi_\alpha \circ \varphi_\beta^{-1}\colon \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$

is a smooth ($=C^\infty$) map between open subsets of $\mathbb{R}^k$.

Explain how to find a topology on M satisfying (M1), (M2) and (M3) in Definition IV.1.1. In general, this topology will not be Hausdorff (cf. Exercise IV.1.5 below). ■

Exercise IV.1.5. We discuss an example of a "non-Hausdorff manifold". We endow the set $S := (\{1\} \times \mathbb{R}) \cup (\{2\} \times \mathbb{R}) \subseteq \mathbb{R}^2$ with the subspace topology and define an equivalence relation on S by

$$(1, x) \sim (2, y) \quad \iff \quad x = y \neq 0,$$

so that all classes except $[1, 0]$ and $[2, 0]$ contain 2 points. The topological quotient space

$$M := S/\sim = \{[1, x] \colon x \in \mathbb{R}\} \cup \{[2, 0]\} = \{[2, x] \colon x \in \mathbb{R}\} \cup \{[1, 0]\}$$

is the union of a real line with an extra point, but the two points $[1, 0]$ and $[2, 0]$ have no disjoint open neighborhoods.

The subsets $U_j := \{[j, x] \colon x \in \mathbb{R}\}$, $j = 1, 2$, of M are open, and the maps

$$\varphi_j \colon U_j \to \mathbb{R}, \quad [j, x] \mapsto x,$$

are homeomorphism defining a smooth atlas on M. ■

IV.2. Some useful techniques

In this subsection we describe some methods to construct Lie group structures on groups, starting from a manifold structure on some "identity neighborhood" for which the group operations are smooth close to $\mathbf{1}$.

Group topologies from local data

The following lemma describes how to construct a group topology on a group from a filter basis of subsets which then becomes a filter basis of identity neighborhoods for the group topology.

Definition IV.2.1. (a) Let X be a set. A subset $\mathcal{F} \subseteq \mathbb{P}(X)$ of subsets of X is called a *filter basis* if the following axioms are satisfied:
(F1) $\mathcal{F} \neq \emptyset$.
(F2) Each set $F \in \mathcal{F}$ is non-empty.
(F3) $A, B \in \mathcal{F} \Rightarrow (\exists C \in \mathcal{F})\ C \subseteq A \cap B$.

Lemma IV.2.2. *Let G be a group and $\mathcal{F}$ a filter basis of subsets of G satisfying $\bigcap \mathcal{F} = \{\mathbf{1}\}$ and*

(U1) $(\forall U \in \mathcal{F})(\exists V \in \mathcal{F})\ VV \subseteq U$.

(U2) $(\forall U \in \mathcal{F})(\exists V \in \mathcal{F})\ V^{-1} \subseteq U$.

(U3) $(\forall U \in \mathcal{F})(\forall g \in G)(\exists V \in \mathcal{F})\ gVg^{-1} \subseteq U$.

Then there exists a unique group topology on G such that $\mathcal{F}$ is a basis of $\mathbf{1}$-neighborhoods in G. This topology is given by

$$\{U \subseteq G\colon (\forall g \in U)(\exists V \in \mathcal{F})\ gV \subseteq U\}.$$

Proof. Let

$$\tau := \{U \subseteq G\colon (\forall g \in U)(\exists V \in \mathcal{F})\ gV \subseteq U\}.$$

First we show that τ is a topology. Clearly $\emptyset, G \in \tau$. Let $(U_j)_{j\in J}$ be a family of elements of τ and $U := \bigcup_{j\in J} U_j$. For each $g \in U$ exists a $j_0 \in J$ with $g \in U_j$ and a $V \in \mathcal{F}$ with $gV \subseteq U_{j_0} \subseteq U$. Thus $U \in \tau$ and we see that τ is stable under arbitrary unions.

If $U_1, U_2 \in \tau$ and $g \in U_1 \cap U_2$, there exist $V_1, V_2 \in \mathcal{F}$ with $gV_i \subseteq U_i$. Since $\mathcal{F}$ is a filter basis, there exists $V_3 \in \mathcal{F}$ with $V_3 \subseteq V_1 \cap V_2$, and then $gV_3 \subseteq U_1 \cap U_2$. We conclude that $U_1 \cap U_2 \in \tau$, and hence that τ is a topology on G.

We claim that the interior U^0 of a subset $U \subseteq G$ is given by

$$U_1 := \{u \in U\colon (\exists V \in \mathcal{F})\ uV \subseteq U\}.$$

In fact, if there exists a $V \in \mathcal{F}$ with $uV \subseteq U$, then we pick a $W \in \mathcal{F}$ with $WW \subseteq V$ and obtain $uWW \subseteq U$, so that $uW \subseteq U_1$. Hence U_1 is open, and it clearly is the largest open subset contained in U, i.e., $U_1 = U^0$. It follows in particular that U is a neighborhood of g if and only if $g \in U^0$, and we see in particular that $\mathcal{F}$ is a basis of the neighborhood filter of $\mathbf{1}$. The property $\bigcap \mathcal{F} = \{\mathbf{1}\}$ implies that for $x \neq y$ there exists $U \in \mathcal{F}$ with $y^{-1}x \notin U$. For $V \in \mathcal{F}$ with $VV \subseteq U$ and $W \in \mathcal{F}$ with $W^{-1} \subseteq V$ we then obtain $y^{-1}x \notin VW^{-1}$, i.e., $xW \cap yV = \emptyset$. Thus (G, τ) is a Hausdorff space.

To see that G is a topological group, we have to verify that the map

$$f\colon G \times G \to G, \quad (x, y) \mapsto xy^{-1}$$

is continuous. So let $x, y \in G$, $U \in \mathcal{F}$ and pick $V \in \mathcal{F}$ with $yVy^{-1} \subseteq U$ and $W \in \mathcal{F}$ with $WW^{-1} \subseteq V$. Then

$$f(xW, yW) = xWW^{-1}y^{-1} = xy^{-1}y(WW^{-1})y^{-1} \subseteq xy^{-1}yVy^{-1} \subseteq xy^{-1}U,$$

implies that f is continuous in (x, y). ∎

Before we turn to Lie group structures, it is illuminating to first consider the topological variant of the following theorem.

Lemma IV.2.3. *Let G be a group and $U = U^{-1}$ a symmetric subset containing $\mathbf{1}$. We further assume that U carries a Hausdorff topology for which*

(T1) *$D := \{(x,y) \in U \times U : xy \in U\}$ is an open subset and the group multiplication $m_U: D \to U, (x,y) \mapsto xy$ is continuous,*

(T2) *the inversion map $\eta_U: U \to U, u \mapsto u^{-1}$ is continuous, and*

(T3) *for each $g \in G$, there exists an open $\mathbf{1}$-neighborhood U_g in U with $c_g(U_g) \subseteq U$, such that the conjugation map*

$$c_g: U_g \to U, \quad x \mapsto gxg^{-1}$$

is continuous.

Then there exists a unique group topology on G for which the inclusion map $U \hookrightarrow G$ is a homeomorphism onto an open subset of G.

If, in addition, U generates G, then (T1/2) *imply* (T3).

Proof. First we consider the filter basis $\mathcal{F}$ of $\mathbf{1}$-neighborhoods in U. Then (T1) implies (U1), (T2) implies (U2), and (T3) implies (U3). Moreover, the assumption that U is Hausdorff implies that $\bigcap \mathcal{F} = \{\mathbf{1}\}$. Therefore Lemma IV.2.2 implies that G carries a unique structure of a (Hausdorff) topological group for which $\mathcal{F}$ is a basis of $\mathbf{1}$-neighborhoods.

We claim that the inclusion map $U \to G$ is an open embedding. So let $x \in U$. Then

$$U_x := U \cap x^{-1}U = \{y \in U : (x,y) \in D\}$$

is open in U and λ_x restricts to a continuous map $U_x \to U$ with image $U_{x^{-1}}$. Its inverse is also continuous. Hence $\lambda_x^U: U_x \to U_{x^{-1}}, y \mapsto xy$ is a homeomorphism. We conclude that a basis for the neighborhood filter of $x \in U$ consists of sets of the form xV, V a neighborhood of $\mathbf{1}$, and hence that the inclusion map $U \hookrightarrow G$ is an open embedding.

Suppose, in addition, that G is generated by U. For each $g \in U$, there exists an open $\mathbf{1}$-neighborhood U_g with $gU_g \times \{g^{-1}\} \subseteq D$. Then $c_g(U_g) \subseteq U$, and the continuity of m_U implies that $c_g|_{U_g}: U_g \to U$ is continuous.

Hence, for each $g \in U$, the conjugation c_g is continuous in a neighborhood of $\mathbf{1}$. Since the set of all these g is a submonoid of G containing U, it contains U^n for each $n \in \mathbb{N}$, hence all of G because G is generated by $U = U^{-1}$. Therefore (T3) follows from (T1) and (T2). ■

The following theorem, the smooth version of the preceding lemma, is an important tool to construct Lie group structures on groups.

Theorem IV.2.4. *Let G be a group and $U = U^{-1}$ a symmetric subset containing $\mathbf{1}$. We further assume that U is a smooth manifold and that*

(L1) *$D := \{(x,y) \in U \times U : xy \in U\}$ is an open subset and the multiplication $m_U: D \to U, (x,y) \mapsto xy$ is smooth,*

(L2) *the inversion map $\eta_U: U \to U, u \mapsto u^{-1}$ is smooth, and*

(L3) *for each $g \in G$ there exists an open $\mathbf{1}$-neighborhood $U_g \subseteq U$ with $c_g(U_g) \subseteq U$ and such that the conjugation map*

$$c_g: U_g \to U, \quad x \mapsto gxg^{-1}$$

is smooth.

Then there exists a unique structure of a Lie group on G such that the inclusion map $U \hookrightarrow G$ is a diffeomorphism onto an open subset of G.

If, in addition, U generates G, then (L1/2) *imply* (L3).

Proof. From the preceding Lemma IV.2.3, we immediately obtain the unique group topology on G for which the inclusion map $U \hookrightarrow G$ is an open embedding.

Now we turn to the manifold structure. Let $V = V^{-1} \subseteq U$ be an open $\mathbf{1}$-neighborhood with $VV \times VV \subseteq D$, for which there exists an $\mathbb{R}^n$-chart (φ, V) of U. For $g \in G$ we consider the maps

$$\varphi_g : gV \to \mathbb{R}^n, \quad \varphi_g(x) = \varphi(g^{-1}x)$$

which are homeomorphisms of gV onto $\varphi(V) \subseteq \mathbb{R}^n$. We claim that $(\varphi_g, gV)_{g\in G}$ is a smooth atlas of G.

Let $g_1, g_2 \in G$ and put $W := g_1V \cap g_2V$. If $W \neq \emptyset$, then $g_2^{-1}g_1 \in VV^{-1} = VV$. The smoothness of the map

$$\psi := \varphi_{g_2} \circ \varphi_{g_1}^{-1}|_{\varphi_{g_1}(W)} : \varphi_{g_1}(W) \to \varphi_{g_2}(W)$$

given by

$$\psi(x) = \varphi_{g_2}(\varphi_{g_1}^{-1}(x)) = \varphi_{g_2}(g_1\varphi^{-1}(x)) = \varphi(g_2^{-1}g_1\varphi^{-1}(x))$$

follows from the smoothness of the multiplication $VV \times VV \to U$. This proves that the charts $(\varphi_g, gU)_{g\in G}$ form a smooth atlas of G. Moreover, the construction implies that all left translations of G are smooth maps.

The construction also shows that for each $g \in G$ the conjugation $c_g : G \to G$ is smooth in a neighborhood of $\mathbf{1}$. Since all left translations are smooth, and

$$c_g \circ \lambda_x = \lambda_{c_g(x)} \circ c_g,$$

the smoothness of c_g in a neighborhood of $x \in G$ follows. Therefore all conjugations and hence also all right multiplications are smooth. The smoothness of the inversion follows from its smoothness on V and the fact that left and right multiplications are smooth. Finally, the smoothness of the multiplication follows from the smoothness in $\mathbf{1} \times \mathbf{1}$ because

$$g_1xg_2y = g_1g_2c_{g_2^{-1}}(x)y.$$

Next we show that the inclusion $U \hookrightarrow G$ of U is a diffeomorphism. So let $x \in U$ and recall $U_{x^{-1}} := U \cap xU$. We have seen above that $\lambda^U_{x^{-1}} : U_{x^{-1}} \to U_x$ is a diffeomorphism of the open neighborhood $U_{x^{-1}}$ of x onto the open $\mathbf{1}$-neighborhood U_x in U. Since $\lambda_x : G \to G$ is a diffeomorphism, the inclusion $\lambda_x \circ \lambda^U_{x^{-1}} : U_{x^{-1}} \to G$ is a diffeomorphism. As x was arbitrary, the inclusion of U in G is a diffeomorphic embedding.

The uniqueness of the Lie group structure is clear because each locally diffeomorphic bijective homomorphism between Lie groups is a diffeomorphism (Lemma IV.1.15).

Finally, we assume that G is generated by U. We show that in this case (L3) is a consequence of (L1) and (L2); the argument is similar to the topological case. Indeed, for each $g \in U$, there exists an open $\mathbf{1}$-neighborhood U_g with $gU_g \times \{g^{-1}\} \subseteq D$. Then $c_g(U_g) \subseteq U$, and the smoothness of m_U implies that $c_g|_{U_g}: U_g \to U$ is smooth. Hence, for each $g \in U$, the conjugation c_g is smooth in a neighborhood of $\mathbf{1}$. Since the set of all g with this property is a submonoid of G containing U, it contains U^n for each $n \in \mathbb{N}$, hence all of G because G is generated by $U = U^{-1}$. Therefore (L3) is satisfied. ■

Corollary IV.2.5. *Let G be a group and $N \trianglelefteq G$ a normal subgroup of G that carries a Lie group structure. Then there exists a Lie group structure on G for which N is an open subgroup if and only if for each $g \in G$ the restriction $c_g|_N$ is a smooth automorphism of N.*

Proof. If N is an open normal subgroup of the Lie group G, then clearly all inner automorphisms of G restrict to smooth automorphisms of N.

Suppose, conversely, that N is a normal subgroup of the group G which is a Lie group and that all inner automorphisms of G restrict to smooth automorphisms of N. Then we can apply Theorem IV.2.4 with $U = N$ and obtain a Lie group structure on G for which the inclusion $N \to G$ is a diffeomorphism onto an open subgroup of G. ■

We call a surjective morphism $\varphi: G \to H$ of topological groups a *covering* if it is an open map with discrete kernel. It can be shown that this is equivalent to being a covering in the standard topological sense (cf. Example V.2.2 below).

Corollary IV.2.6. *Let $\varphi: G \to H$ be a covering of topological groups. If G or H is a Lie group, then the other group has a unique Lie group structure for which φ is a morphism of Lie groups which is a local diffeomorphism.*

Proof. Since $\ker \varphi$ is discrete, there exists an open symmetric identity neighborhood $U_G \subseteq G$ for which $U_G^3 := U_G U_G U_G$ intersects $\ker(\varphi)$ in $\{\mathbf{1}\}$. For $x, y \in U_G$ with $\varphi(x) = \varphi(y)$ we then have $x^{-1}y \in U_G^2 \cap \ker(\varphi) = \{\mathbf{1}\}$, so that $\varphi|_{U_G}$ is injective. Since φ is an open map, this implies that $\varphi|_{U_G}$ is a homeomorphism onto an open subset $U_H := \varphi(U_G)$ of H.

Suppose first that G is a Lie group. Then we apply Theorem IV.2.4 to U_H, endowed with the manifold structure for which $\varphi|_{U_G}$ is a diffeomorphism. Then (L2) follows from $\varphi(x)^{-1} = \varphi(x^{-1})$. To verify the smoothness of the multiplication map

$$m_{U_H}: D_H := \{(a,b) \in U_H \times U_H : ab \in U_H\} \to U_H,$$

we first observe that, if $x, y \in U_G$ satisfy $(\varphi(x), \varphi(y)) \in D_H$, i.e., $\varphi(xy) \in U_H$, then there exists a $z \in U_G$ with $\varphi(xy) = \varphi(z)$, and $xyz^{-1} \in {U_G}^3 \cap \ker(\varphi) = \{\mathbf{1}\}$ yields $z = xy$. We thus have $D_H = (\varphi \times \varphi)(D_G)$ for

$$D_G := \{(x,y) \in U_G \times U_G : xy \in U_G\}$$

and the smoothness of m_{U_H} follows from the smoothness of the multiplication $m_{U_G}\colon D_G \to U_G$ and

$$m_{U_H} \circ (\varphi \times \varphi) = \varphi \circ m_{U_G}.$$

To verify (L3), we note that the surjectivity of φ implies that for each $h \in H$ there is an element $g \in G$ with $\varphi(g) = h$. Now we choose an open $\mathbf{1}$-neighborhood $U_g \subseteq U_G$ with $c_g(U_g) \subseteq U_G$ and put $U_h := \varphi(U_g)$.

If, conversely, H is a Lie group, then we apply Theorem IV.2.4 to U_G, endowed with the manifold structure for which $\varphi|_{U_G}$ is a diffeomorphism onto U_H. Again, (L2) follows right away, and (L1) follows from $(\varphi \circ \varphi)(D_G) \subseteq D_H$ and the smoothness of

$$m_{U_H} \circ (\varphi \circ \varphi) = \varphi \circ m_{U_G}.$$

For (L3), we choose U_g as any open $\mathbf{1}$-neighborhood in U_G with $c_g(U_g) \subseteq U_G$. Then the smoothness of $c_g|_{U_g}$ follows from the smoothness of $\varphi \circ c_g = c_{\varphi(g)} \circ \varphi$. ■

The main point in the local approach is that it emphasizes that the whole differentiable structure is determined by the manifold structure on some neighborhood of the identity. This is very convenient to construct Lie group structures.

Proposition IV.2.7. *Every linear Lie group carries a Lie group structure for which the exponential function* $\exp_G\colon \mathbf{L}(G) \to G$ *is smooth and there exists an open* 0*-neighborhood* $V \subseteq \mathbf{L}(G)$ *on which the Dynkin series converges and* $\exp_G|_V\colon V \to \exp_G(V)$ *is a diffeomorphism onto an open subset of* G *with*

$$\exp_G(x * y) = \exp_G(x)\exp_G(y) \quad \text{for} \quad x, y \in V.$$

Proof. Let $G \subseteq \mathrm{GL}_n(\mathbb{R})$ be a linear Lie group. Then we know from Theorem III.3.1 that there exists an open symmetric 0-neighborhood $W \subseteq \mathbf{L}(G)$ such that

$$\varphi := \exp_G|_W\colon W \to \exp_G(W)$$

is a homeomorphism onto an open $\mathbf{1}$-neighborhood in G. Since $\exp_G\colon \mathbf{L}(G) \to G$ is continuous, there exists an open 0-neighborhood $V \subseteq W$ with

$$\exp_G V \exp_G V \subseteq \exp_G W$$

and $\|x\| < \frac{1}{2}\log(2 - \frac{\sqrt{2}}{2})$ for $x \in V$, so that the Dynkin series for $x * y$ converges for $x, y \in V$ and satisfies

$$\exp_G(x * y) = \exp_G(x)\exp_G(y)$$

(Proposition II.4.5). We also know from Proposition II.4.5 that $x * y$ defines a smooth function on $V \times V \to \mathbf{L}(G)$ because it is the restriction of a smooth function on an open 0-neighborhood of $M_n(\mathbb{R})$.

Since for each $g \in G$ the linear automorphism $\mathrm{Ad}(g)$ of $\mathbf{L}(G)$ is continuous, there exists an open 0-neighborhood $V_g \subseteq \mathbf{L}(G)$ with $\mathrm{Ad}(g)V_g \subseteq V$, and then $\mathrm{Ad}(g)$ restrict to a smooth map

$$\mathrm{Ad}(g)\colon V_g \to V, \quad x \mapsto gxg^{-1}$$

with $\varphi(\mathrm{Ad}(g)x) = g\varphi(x)g^{-1}$.

Therefore $U := \exp_G V = \varphi(V)$, endowed with the manifold structure inherited from V via φ satisfies all assumptions of Theorem IV.2.4, so that we obtain a Lie group structure on G for which $\varphi = \exp_G|_V$ is a diffeomorphism onto the open subset $\exp_G(V)$ of G.

Finally, the smoothness of the exponential function follows from its smoothness on V, $\bigcup_{m\in\mathbb{N}} mV = \mathbf{L}(G)$, and $\exp_G(mx) = \exp_G(x)^m$. ■

If $\mathfrak{g} \subseteq M_n(\mathbb{R})$ is the Lie algebra of a linear Lie group G, i.e., $\langle \exp \mathfrak{g}\rangle$ is a closed subgroup of $\mathrm{GL}_n(\mathbb{R})$, then the preceding proposition shows that $\langle \exp \mathfrak{g}\rangle = G_0$ (Exercise IV.1.2) carries a Lie group structure. A closer inspection of the preceding proof shows that the same technique yields the following sharper result which even applies to groups such as the dense wind.

Proposition IV.2.8. *Let $\mathfrak{g} \subseteq M_n(\mathbb{R})$ be a Lie subalgebra. Then the subgroup $G := \langle \exp \mathfrak{g}\rangle$ of $\mathrm{GL}_n(\mathbb{R})$ generated by $\exp(\mathfrak{g})$ carries a Lie group structure for which there exists an open 0-neighborhood $V \subseteq \mathfrak{g}$ on which the Dynkin series converges and*

$$\exp_G\colon \mathfrak{g} \to G, \quad x \mapsto \exp x$$

maps V diffeomorphism onto its open image in G and satisfies

$$\exp_G(x * y) = \exp_G(x)\exp_G(y) \quad \textit{for} \quad x, y \in V.$$

Proof. Let

$$V := \Big\{x \in \mathfrak{g}\colon \|x\| < \frac{1}{2}\log(2 - \frac{\sqrt{2}}{2})\Big\},$$

so that the Dynkin series for $x * y$ converges for $x, y \in V$ and satisfies

$$\exp(x * y) = \exp(x)\exp(y)$$

(Proposition II.4.5). We also know from Proposition II.4.5 that $x * y$ defines a smooth function on $V \times V \to \mathfrak{g}$ because it is the restriction of a smooth function on an open 0-neighborhood of $M_n(\mathbb{R})$.

We consider the subset $U := \exp(V) \subseteq G$. From $V = -V$ we derive $U = U^{-1}$, and since $\|x\| < \log 2$ for $x \in V$, Proposition II.3.2 implies that $\varphi := \exp|_V$ is injective. We may thus endow U with the manifold structure turning φ into a diffeomorphism.

Then

$$\widetilde{D} = \{(x, y) \in V \times V\colon x * y \in V\}$$

is an open subset of $V \times V$ on which the BCH multiplication is smooth, so that the multiplication $D \to U$ is also smooth. We further observe that

$$\exp(-x) = \exp(x)^{-1},$$

from which it follows that the inversion on U is smooth.

To verify (L3), we first note that for each $x \in \mathfrak{g}$ we have $\mathrm{Ad}(\exp x)\mathfrak{g} = e^{\mathrm{ad}\, x}\mathfrak{g} \subseteq \mathfrak{g}$, from which it follows that $\mathrm{Ad}(g)\mathfrak{g} = \mathfrak{g}$ for each $g \in G$. Hence $\mathrm{Ad}(g)$ induces a linear automorphism of $\mathfrak{g}$, hence a continuous map, so that there exists an open 0-neighborhood $V_g \subseteq \mathfrak{g}$ with $\mathrm{Ad}(g)V_g \subseteq V$. Now $\mathrm{Ad}(g)$ restrict to a smooth map

$$\mathrm{Ad}(g)\colon V_g \to V, \quad x \mapsto gxg^{-1}$$

with $\varphi(\mathrm{Ad}(g)x) = g\varphi(x)g^{-1}$.

Therefore U satisfies all assumptions of Theorem IV.2.4, so that we obtain a Lie group structure on G for which φ, resp., exp induces a local diffeomorphism in 0. ■

Remark IV.2.9. The example of the dense wind shows that we cannot expect that the group $G = \langle \exp \mathfrak{g} \rangle$ is closed in $\mathrm{GL}_n(\mathbb{R})$ or that the inclusion map $G \to \mathrm{GL}_n(\mathbb{R})$ (which is a smooth homomorphism) is a topological embedding. ■

Remark IV.2.10. The concept of a Lie group discussed in this section describes Lie groups as topological groups with some additional structure, given by a homeomorphism of an open $\mathbf{1}$-neighborhood onto some open subset of $\mathbb{R}^n$, so that the group operations in these coordinates are smooth near the identity. This point of view can easily be generalized to infinite-dimensional Lie groups, where $\mathbb{R}^n$ is replaced by a locally convex vector space E. Once it is clear what a smooth map between two open subsets of such spaces is, we can define the concept of an infinite-dimensional Lie group as above, and Theorem IV.2.4 remains true in this context. Important classes of infinite-dimensional Lie groups are Banach- and Fréchet–Lie groups, where E is a Banach space, resp., a Fréchet space. ■

V. Basic covering theory

We have already seen that each morphism of (linear) Lie groups

$$\varphi: G \to H$$

induces a morphism of Lie algebras

$$\mathbf{L}(\varphi): \mathbf{L}(G) \to \mathbf{L}(H)$$

and that φ is uniquely determined on the identity component G_0 of G by $\mathbf{L}(\varphi)$. It is a key problem in Lie theory to determine to which extend we can invert this process, i.e., integrate morphisms of Lie algebras $\mathbf{L}(G) \to \mathbf{L}(H)$ to morphisms of Lie groups. If G is connected, the condition we encounter is that G should be simply connected (defined below), and if this is not the case, one can always pass to its simply connected covering group $\widetilde{G}$ and integrate to a morphism $\widetilde{G} \to H$. In the present chapter we develop the topological and group theoretic background needed to deal with these problems.

V.1. The fundamental group

To define the notion of a simply connected space, we need its fundamental group. The elements of this group are homotopy classes of loops. The present section develops this concept and provides some of its basic properties.

Definition V.1.1. Let X be a topological space, $I := [0,1]$, and $x_0 \in X$. We write

$$P(X, x_0) := \{\gamma \in C(I, X): \gamma(0) = x_0\}$$

and

$$P(X, x_0, x_1) := \{\gamma \in P(X, x_0): \gamma(1) = x_1\}.$$

We call two paths $\alpha_0, \alpha_1 \in P(X, x_0, x_1)$ *homotopic*, written $\alpha_0 \sim \alpha_1$, if there exists a continuous map

$$H: I \times I \to X \quad \text{with} \quad H_0 = \alpha_0, \quad H_1 = \alpha_1$$

and

$$(\forall t \in I) \quad H(t,0) = x_0,\ H(t,1) = x_1.$$

It is easy to show that $\sim$ is an equivalence relation (Exercise V.1.2), called *homotopy*. The homotopy class of α is denoted $[\alpha]$.

We write $\Omega(X, x_0) := P(X, x_0, x_0)$ for the set of loops in x_0. For $\alpha \in P(X, x_0, x_1)$ and $\beta \in P(X, x_1, x_2)$ we define a product $\alpha * \beta \in P(X, x_0, x_2)$ by

$$(\alpha * \beta)(t) := \begin{cases} \alpha(2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1) & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$ ■

Lemma V.1.2. *If $\varphi: [0,1] \to [0,1]$ is a continuous map with $\varphi(0) = 0$ and $\varphi(1) = 1$, then for each $\alpha \in P(X, x_0, x_1)$ we have $\alpha \sim \alpha \circ \varphi$.*

Proof. $H(t,s) := \alpha(t + (1-t)\varphi(s))$. ■

Proposition V.1.3. *The following assertions hold:*

(1) *$\alpha_1 \sim \alpha_2$ and $\beta_1 \sim \beta_2$ implies $\alpha_1 * \beta_1 \sim \alpha_2 * \beta_2$, so that we obtain a well-defined product*

$$[\alpha] * [\beta] := [\alpha * \beta]$$

of homotopy classes.

(2) *If $x_0 \in \Omega(X, x_0)$ denotes the constant map $I \to X$, then*

$$[x_0] * [\alpha] = [\alpha] = [\alpha] * [x_1] \quad \textit{for} \quad \alpha \in P(X, x_0, x_1).$$

(3) *(Associativity) $[\alpha * \beta] * [\gamma] = [\alpha] * [\beta * \gamma]$ for $\alpha \in P(X, x_0, x_1)$, $\beta \in P(X, x_1, x_2)$ and $\gamma \in P(X, x_2, x_3)$.*

(4) *(Inverse) For $\alpha \in P(X, x_0, x_1)$ and $\overline{\alpha}(t) := \alpha(1-t)$ we have*

$$[\alpha] * [\overline{\alpha}] = [x_0].$$

(5) *(Functoriality) For any continuous map $\varphi: X \to Y$ with $\varphi(x_0) = y_0$ we have*

$$(\varphi \circ \alpha) * (\varphi \circ \beta) = \varphi \circ (\alpha * \beta).$$

Proof. (1) If H^α is a homotopy from α_1 to α_2 and H^β a homotopy from β_1 to β_2, then we put

$$H(t,s) := \begin{cases} H^\alpha(t, 2s) & \text{for } 0 \leq s \leq \frac{1}{2} \\ H^\beta(t, 2s-1) & \text{for } \frac{1}{2} \leq s \leq 1 \end{cases}$$

(cf. Exercise V.1.1).

(2) For the first assertion we use Lemma V.1.2 and

$$x_0 * \alpha = \alpha \circ \varphi \quad \text{for} \quad \varphi(t) := \begin{cases} 0 & \text{for } 0 \leq t \leq \frac{1}{2} \\ 2t-1 & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

For the second we have

$$\alpha * x_1 = \alpha \circ \varphi \quad \text{for} \quad \varphi(t) := \begin{cases} 2t & \text{for } 0 \leq t \leq \frac{1}{2} \\ 1 & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

(3) We have $(\alpha * \beta) * \gamma = (\alpha * (\beta * \gamma)) \circ \varphi$ for

$$\varphi(t) := \begin{cases} 2t & \text{for } 0 \le t \le \frac{1}{4} \\ \frac{1}{4} + t & \text{for } \frac{1}{4} \le t \le \frac{1}{2} \\ \frac{t+1}{2} & \text{for } \frac{1}{2} \le t \le 1. \end{cases}$$

(4)

$$H(t,s) := \begin{cases} \alpha(2s) & \text{for } s \le \frac{1-t}{2} \\ \alpha(1-t) & \text{for } \frac{1-t}{2} \le s \le \frac{1+t}{2} \\ \overline{\alpha}(2s-1) & \text{for } s \ge \frac{1+t}{2}. \end{cases}$$

(5) is trivial. ■

Definition V.1.4. From the preceding definition we derive in particular that the set

$$\pi_1(X, x_0) := \Omega(X, x_0)/\sim$$

of homotopy classes of loops in x_0 carries a natural group structure. This group is called the *fundamental group of X with respect to x_0*.

A pathwise connected space X is called *simply connected* if $\pi_1(X, x_0)$ vanishes for some $x_0 \in X$ (which implies that it vanishes for each $x_0 \in X$; Exercise V.1.4). ■

Lemma V.1.5. (Functoriality of the fundamental group) *If $f: X \to Y$ is a continuous map with $f(x_0) = y_0$, then*

$$\pi_1(f): \pi_1(X, x_0) \to \pi_1(Y, y_0), \quad [\gamma] \mapsto [f \circ \gamma]$$

is a group homomorphism. Moreover, we have

$$\pi_1(\mathrm{id}_X) = \mathrm{id}_{\pi_1(X,x_0)} \quad \text{and} \quad \pi_1(f \circ g) = \pi_1(f) \circ \pi_1(g).$$

Proof. This follows directly from Proposition V.1.3(v). ■

Remark V.1.6. The map

$$\sigma: \pi_1(X, x_0) \times P(X, x_0)/\sim \to P(X, x_0)/\sim, \quad ([\alpha], [\beta]) \mapsto [\alpha * \beta] = [\alpha] * [\beta]$$

defines an action of the group $\pi_1(X, x_0)$ on the set $P(X, x_0)/\sim$ of homotopy classes of paths starting in x_0 (Lemma V.1.3). ■

Remark V.1.7. (a) Suppose that the topological space X is contractible, i.e., there exists a continuous map $H: I \times X \to X$ and $x_0 \in X$ with $H(0, x) = x$ and $H(1, x) = x_0$ for $x \in X$. Then $\pi_1(X, x_0) = \{[x_0]\}$ is trivial (Exercise).

(b) $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

(c) $\pi_1(\mathbb{R}^n, 0) = \{0\}$ because $\mathbb{R}^n$ is contractible.

More generally, if the open subset $\Omega \subseteq \mathbb{R}^n$ is starlike with respect to x_0, then $H(t,x) := x + t(x - x_0)$ yields a contraction to x_0, and we conclude that $\pi_1(\Omega, x_0) = \{\mathbf{1}\}$.
(d) If $G \subseteq \mathrm{GL}_n(\mathbb{R})$ is a linear Lie group with a polar decomposition, i.e., for $K := G \cap \mathrm{O}_n(\mathbb{R})$ and $\mathfrak{p} := \mathbf{L}(G) \cap \mathrm{Sym}_n(\mathbb{R})$, the polar map

$$p\colon K \times \mathfrak{p} \to G, \quad (k,x) \mapsto ke^x$$

is a homeomorphism, then the inclusion $K \to G$ induces an isomorphism

$$\pi_1(K, \mathbf{1}) \to \pi_1(G, \mathbf{1})$$

because the vector space $\mathfrak{p}$ is contractible. ■

The following lemma implies in particular, that fundamental groups of topological groups are always abelian.

Lemma V.1.8. *Let G be a topological group and consider the identity element $\mathbf{1}$ as a base point. Then the path space $P(G, \mathbf{1})$ also carries a natural group structure given by the pointwise product $(\alpha \cdot \beta)(t) := \alpha(t)\beta(t)$ and we have*
(1) *$\alpha \sim \alpha'$, $\beta \sim \beta'$ implies $\alpha \cdot \beta \sim \alpha' \cdot \beta'$, so that we obtain a well-defined product*

$$[\alpha] \cdot [\beta] := [\alpha \cdot \beta]$$

of homotopy classes, defining a group structure on $P(G, \mathbf{1})/\sim$.
(2) *$\alpha \sim \beta \iff \alpha\beta^{-1} \sim \mathbf{1}$, the constant map.*
(3) *(Commutativity) $[\alpha] \cdot [\beta] = [\beta] \cdot [\alpha]$ for $\alpha, \beta \in \Omega(G, \mathbf{1})$.*
(4) *(Consistency) $[\alpha] \cdot [\beta] = [\alpha] * [\beta]$ for $\alpha \in \Omega(G, \mathbf{1})$, $\beta \in P(G, \mathbf{1})$.*

Proof. (1) follows by composing homotopies with the product map m_G.
(2) follows from (1).
(3)

$$[\alpha][\beta] = [\alpha * \mathbf{1}][\mathbf{1} * \beta] = [(\alpha * \mathbf{1})(\mathbf{1} * \beta)] = [(\mathbf{1} * \beta)(\alpha * \mathbf{1})] = [\mathbf{1} * \beta][\alpha * \mathbf{1}] = [\beta][\alpha].$$

(4) $[\alpha][\beta] = [(\alpha * \mathbf{1})(\mathbf{1} * \beta)] = [\alpha * \beta] = [\alpha] * [\beta]$. ■

As a consequence of (4), we can calculate the product of homotopy classes as a pointwise product of representatives and obtain:

Proposition V.1.9. *For each topological group G, the fundamental group $\pi_1(G) := \pi_1(G, \mathbf{1})$ is abelian.*

Proof. We only have to combine (3) and (4) in Lemma V.1.8 for loops $\alpha, \beta \in \Omega(G, \mathbf{1})$. ■

Exercises for Section V.1.

Exercise V.1.1. If $f: X \to Y$ is a map between topological spaces and $X = X_1 \cup \ldots \cup X_n$ holds with closed subsets $X_1, \ldots, X_n$, then f is continuous if and only if all restrictions $f|_{X_i}$ are continuous. ■

Exercise V.1.2. Show that the homotopy relation on $P(X, x_0, x_1)$ is an equivalence relation. Hint: Exercise V.1.1 helps to glue homotopies. ■

Exercise V.1.3. Show that for $n > 1$ the sphere $\mathbb{S}^n$ is simply connected. For the proof use the following steps:
(a) Let $\gamma : [0,1] \to \mathbb{S}^n$ be continuous. Then there exists an $m > 0$ such that $\|\gamma(t) - \gamma(t')\| < \frac{1}{2}$ for $|t - t'| < \frac{1}{m}$.
(b) Define $\widetilde{\alpha} : [0,1] \to \mathbb{R}^{n+1}$ as the piecewise affine curve with $\widetilde{\alpha}(\frac{k}{m}) = \gamma(\frac{k}{m})$ for $k = 0, \ldots, m$. Then $\alpha(t) := \frac{1}{\|\widetilde{\alpha}(t)\|}\widetilde{\alpha}(t)$ defines a continuous curve $\alpha : [0,1] \to \mathbb{S}^n$.
(c) $\alpha \sim \gamma$. Hint: Consider

$$F(t,s) := \frac{(1-s)\gamma(t) + s\alpha(t)}{\|(1-s)\gamma(t) + s\alpha(t)\|}.$$

(d) α is not surjective. The image of α is the central projection of a polygonal arc on the sphere.
(e) If $\beta \in \Omega(\mathbb{S}^1, y_0)$ is not surjective, then $\beta \sim y_0$ (it is homotopic to a constant map). Hint: Let $p \in \mathbb{S}^n \setminus \operatorname{im} \beta$. Using stereographic projection, where p corresponds to the point at infinity, show that $\mathbb{S}^n \setminus \{p\}$ is homeomorphic to $\mathbb{R}^n$, hence contractible.
(f) $\pi_1(\mathbb{S}^n, y_0) = \{[y_0]\}$ for $n \geq 2$ and $y_0 \in \mathbb{S}^n$. ■

Exercise V.1.4. Let X be a topological space x_0, x_1 and $\alpha \in P(X, x_0, x_1)$ a path from x_0 to x_1. Show that the map

$$C: \pi_1(X, x_1) \to \pi_1(X, x_0), \quad [\gamma] \mapsto [\alpha * \gamma * \overline{\alpha}]$$

is an isomorphism of groups. In this sense the fundamental group does not depend on the base point if X is arcwise connected. ■

V.2. Coverings

In this section we discuss the concept of a covering map. Its main application in Lie theory is that it provides for each connected Lie group a simply connected covering group and hence also a tool to calculate its fundamental group. In the following chapter we shall investigate to which extent a Lie group is determined by its Lie algebra and its fundamental group.

Definition V.2.1. Let X and Y be topological spaces. A continuous map $q: X \to Y$ is called a *covering* if each $y \in Y$ has an open neighborhood U such that $q^{-1}(U)$ is a non-empty disjoint union of open subsets $(V_i)_{i \in I}$, such that for each $i \in I$ the restriction $q|_{V_i}: V_i \to U$ is a homeomorphism. We call any such U an *elementary* open subset of X.

Note that this condition implies in particular that q is surjective. ∎

Examples V.2.2. (a) The exponential function

$$\exp: \mathbb{C} \to \mathbb{C}^\times, \quad z \mapsto e^z$$

is a covering map.

(b) The map

$$q: \mathbb{R} \to \mathbb{T}, x \mapsto e^{ix}$$

is a covering.

(c) The kth power maps

$$p_k: \mathbb{C}^\times \to \mathbb{C}^\times, \quad z \mapsto z^k$$

are coverings.

(d) If $q: G \to H$ is a surjective open morphism of topological groups with discrete kernel, then q is a covering (Exercise V.2.2). All the examples (a)-(c) are of this type. ∎

Lemma V.2.3. (Lebesgue number) *Let (X, d) be a compact metric space and $(U_i)_{i \in I}$ an open covering. Then there exists a positive number $\lambda > 0$, called the Lebesgue number of the covering, such that any subset $S \subseteq X$ with diameter $\leq \lambda$ is contained in some U_i.*

Proof. Let us assume that such a number λ does not exist. Then there exists for each $n \in \mathbb{N}$ a subset S_n of diameter $\leq \frac{1}{n}$ which is not contained in some U_i. Pick a point $s_n \in S_n$. Then the sequence (s_n) has a subsequence converging to some $s \in X$. Then s is contained in some U_i, and since U_i is open, there exists an $\varepsilon > 0$ with $U_\varepsilon(s) \subseteq U_i$. If $n \in \mathbb{N}$ is such that $\frac{1}{n} < \frac{\varepsilon}{2}$ and $d(s_n, s) < \frac{\varepsilon}{2}$, we arrive at the contradiction $S_n \subseteq U_{\varepsilon/2}(s_n) \subseteq U_\varepsilon(s) \subseteq U_i$. ∎

Remark V.2.4. (1) If $(U_i)_{i \in I}$ is an open cover of the unit interval $[0, 1]$, then there exists an $n > 0$ such that all subsets of the form $\left[\frac{i}{n}, \frac{i+1}{n}\right]$, $i = 0, \ldots, n-1$, are contained in some U_i.

(2) If $(U_i)_{i \in I}$ is an open cover of the unit square $[0, 1]^2$, then there exists an $n > 0$ such that all subsets of the form

$$\left[\frac{i}{n}, \frac{i+1}{n}\right] \times \left[\frac{j}{n}, \frac{j+1}{n}\right], \quad i, j = 0, \ldots, n-1,$$

are contained in some U_i. ∎

Theorem V.2.5. (The Path Lifting Property) *Let $q\colon X \to Y$ be a covering map and $\gamma\colon I = [0,1] \to Y$ a path. Let $x_0 \in X$ be such that $q(x_0) = \gamma(0)$. Then there exists a unique path $\widetilde{\gamma}\colon I \to X$ such that*

$$q \circ \widetilde{\gamma} = \gamma \quad \text{and} \quad \widetilde{\gamma}(0) = x_0.$$

Proof. By the Lebesgue Lemma (Lemma V.2.3), there exists an $n \in \mathbb{N}$ such that all sets $\gamma(\big[\frac{i}{n}, \frac{i+1}{n}\big])$, $i = 0, \ldots, n-1$, are contained in some elementary set U_i. We now use induction to construct $\widetilde{\gamma}$. Let $V_0 \subseteq q^{-1}(U_0)$ be an open subset containing x_0 for which $q\,|_{V_0}$ is a homeomorphism onto U_0 and define $\widetilde{\gamma}$ on $t \in \big[0, \frac{1}{n}\big]$ by

$$\widetilde{\gamma}(t) := (q\,|_{V_0})^{-1} \circ \gamma(t).$$

Assume that we have already construction a continuous lift $\widetilde{\gamma}$ of γ on the interval $\big[0, \frac{k}{n}\big]$ and that $k < n$. Then we pick an open subset $V_k \subseteq X$ containing $\widetilde{\gamma}(\frac{k}{n})$ for which $q\,|_{V_k}$ is a homeomorphims onto U_k and define $\widetilde{\gamma}$ for $t \in \big[\frac{k}{n}, \frac{k+1}{n}\big]$ by

$$\widetilde{\gamma}(t) := (q\,|_{V_k})^{-1} \circ \gamma(t).$$

We thus obtain the required lift $\widetilde{\gamma}$ of γ.

If $\widehat{\gamma}\colon [0,1] \to X$ is any continuous lift of γ with $\widehat{\gamma}(0) = x_0$, then $\widehat{\gamma}(\big[0, \frac{1}{n}\big])$ is a connected subset of $q^{-1}(U_0)$ containing x_0, hence contained in V_0, showing that $\widetilde{\eta}$ coincides with $\widehat{\gamma}$ on $\big[0, \frac{1}{n}\big]$. Applying the same argument at each step of the induction, we obtain $\widehat{\gamma} = \widetilde{\gamma}$, so that the lift $\widetilde{\gamma}$ is uniqe. ■

Theorem V.2.6. (The Covering Homotopy Theorem) *Let $q\colon X \to Y$ be a covering map and $H\colon I^2 \to Y$ be a homotopy with fixed endpoints of the paths $\gamma := H_0$ and $\eta := H_1$. For any lift $\widetilde{\gamma}$ of γ there exists a unique lift $G\colon I^2 \to X$ of H with $G_0 = \widetilde{\gamma}$. Then $\widetilde{\eta} := G_1$ is the unique lift of η starting in the same point as $\widetilde{\gamma}$ and G is a homotopy from $\widetilde{\gamma}$ to $\widetilde{\eta}$. In particular, lifts of homotopic curves in Y starting in the same point are homotopic in X.*

Proof. Using the Path Lifting Property (Theorem V.2.5), we find for each $t \in I$ a unique continuous lift $I \to X, s \mapsto G(s,t)$, starting in $\widetilde{\gamma}(t)$ with $q(G(s,t)) = H(s,t)$. It remains to show that the so obtained map $G\colon I^2 \to X$ is continuous.

So let $s \in I$. Using Lemma V.2.3, we find a natural number n such that for each connected neighborhood U_s of s of diameter $\leq \frac{1}{n}$ and each $i = 0, \ldots, n$, the set $H(U_s \times \big[\frac{i}{n}, \frac{i+1}{n}\big])$ is contained in some elementary subset U_i of Y. Assuming that G is continuous in $U_s \times \{\frac{i}{n}\}$, G maps this set into a connected subset of $q^{-1}(U_i)$, hence into some open subset V_i for which $q\,|_{V_i}$ is a homeomorphism onto U_i. But then the lift G on $U_s \times \big[\frac{i}{n}, \frac{i+1}{n}\big]$ must be contained in V_i, so that it is of the form $(q\,|_{V_i})^{-1} \circ H$, hence continuous. This means that G is continuous on $U_s \times \big[\frac{i}{n}, \frac{i+1}{n}\big]$. Now an inductive argument shows that G is continuous on $U_s \times I$, and hence on the whole square I^2.

Since the fibers of q are discrete and the curves $H(s,0)$ and $H(s,1)$ are constant, the curves $G(s,0)$ and $G(s,1)$ are also constant. Therefore $\widetilde{\eta}$ is the unique lift of η starting in $\widetilde{\gamma}(0) = G(0,0) = G(1,0)$ and G is a homotopy with fixed endpoints from $\widetilde{\gamma}$ to $\widetilde{\eta}$. ■

Corollary V.2.7. *If $q: X \to Y$ is a covering with $q(x_0) = y_0$, then the corresponding homomorphism*

$$\pi_1(q): \pi_1(X, x_0) \to \pi_1(Y, y_0), \quad [\gamma] \mapsto [q \circ \gamma]$$

is injective.

Proof. If γ, η are loops in x_0 with $[q \circ \gamma] = [q \circ \eta]$, then the Covering Homotopy Theorem implies that γ and η are homotopic. Therefore $[\gamma] = [\eta]$ shows that $\pi_1(q)$ is injective. ■

Corollary V.2.8. *If Y is and simply connected and X is arcwise connected, then each covering map $q: X \to Y$ is a homeomorphism.*

Proof. Since q is an open continuous map, it remains to show that q is injective. So pick $x_0 \in X$ and $y_0 \in Y$ with $q(x_0) = y_0$. If $x \in X$ also satisfies $q(x) = y_0$, then there exists a path $\alpha \in P(X, x_0, x)$ from x_0 to x. Now $q \circ \alpha$ is a loop in Y, hence contractible because Y is simply connected. Now the Covering Homotopy Theorem implies that the unique lift α of $q \circ \alpha$ starting in x_0 is a loop, and therefore that $x_0 = x$. This proves that q is injective. ■

The following theorem provides a more powerful tool, from which the preceding corollary easily follows.

Theorem V.2.9. (The Lifting Theorem) *Assume that $q: X \to Y$ is a covering map with $q(x_0) = y_0$, that W is arcwise connected and locally arcwise connected and that $f: W \to Y$ is a given map with $f(w_0) = y_0$. Then a continuous map $g: W \to X$ with*

$$g(w_0) = x_0 \quad \text{and} \quad q \circ g = f$$

exists if and only if

$$(*) \qquad \pi_1(f)(\pi_1(W, w_0)) \subseteq \pi_1(q)(\pi_1(X, x_0)).$$

If it exists, the map g is uniquely determined. Condition $(*)$ is in particular satisfied if W is simply connected.

Proof. If g exists, then $f = q \circ g$ implies that the image of the homomorphism $\pi_1 = \pi_1(q) \circ \pi_1(g)$ is contained in the image of $\pi_1(q)$. Let us, conversely, assume that this condition is satisfied.

To define g, let $w \in W$ and $\alpha_w: I \to W$ be a path from w_0 to w, Then $f \circ \alpha_w: I \to Y$ is a path which has a continuous lift $\beta_w: I \to X$ starting in x_0. We claim that $\beta_w(1)$ does not depend on the choice of the path α_w. Indeed, if α'_w is another path from w_0 to w, then $\alpha_w * \overline{\alpha}'_w$ is a loop in w_0, so that $(f \circ \alpha_w) * (f \circ \overline{\alpha}'_w)$ is a loop in y_0. In view of $(*)$, the homotopy class of this loop is contained in the image of $\pi_1(q)$, so that it has a lift $\eta: I \to X$ which is a loop in x_0. Since the reverse of the second half of η is a lift of $f \circ \alpha'_w$, starting in x_0, it is β'_w, and we obtain

$$\beta'_w(1) = \eta(\frac{1}{2}) = \beta_w(1).$$

We now put $g(w) := \beta_w(1)$, and it remains to see that g is continuous. This is where we shall use the assumption that W is locally arcwise connected. Let $w \in W$ and put $y := f(w)$. Further, let $U \subseteq Y$ be an elementary neighborhood of y and V be an arcwise connected neighborhood of w in W such that $f(V) \subseteq U$. Fix a path α_w from w_0 to w as before. For any point $w' \in W$ we choose a path $\gamma_{w'}$ from w to w' in V, so that $\alpha_w * \gamma_{w'}$ is a path from w_0 to w'. Let $\widetilde{U} \subseteq X$ be an open subset of X for which $q|_{\widetilde{U}}$ is a homeomorphism onto U and $g(w) \in \widetilde{U}$. Then the uniqueness of lifts implies that

$$\beta_{w'} = \beta_w * \big((q|_{\widetilde{U}})^{-1} \circ (f \circ \gamma_{w'})\big).$$

We conclude that

$$g(w') = (q|_{\widetilde{U}})^{-1}(f(w')) \in \widetilde{U},$$

hence that $g|_V$ is continuous.

We finally show that g is unique. In fact, if $h\colon W \to X$ is another lift of f satisfying $h(w_0) = x_0$, then the set $S := \{w \in W\colon g(w) = h(w)\}$ is non-empty and closed. We claim that it is also open. In fact, let $w_1 \in W$ and U be a connected open elementary neighborhood of $f(w_1)$ and V an arcwise connected neighborhood of w_1 with $f(V_1) \subseteq U$. If $\widetilde{U} \subseteq q^{-1}(U)$ is the open subset on which q is a homeomorphism containing $g(w_1) = h(w_1)$, then the arcwise connectedness of V implies that $g(V), h(V) \subseteq \widetilde{U}$, and hence that $V \subseteq S$. Therefore S is open, closed and non-empty, so that the connectedness of W yields $S = W$, i.e., $g = h$. ■

Corollary V.2.10. (Uniqueness of simply connected coverings) *Suppose that Y is locally arcwise connected. If $q_1\colon X_1 \to Y$ and $q_2\colon X_2 \to Y$ are two simply connected arcwise connected coverings, then there exists a homeomorphism $\varphi\colon X_1 \to X_2$ with $q_2 \circ \varphi = q_1$.*

Proof. Since Y is locally arcwise connected, both covering spaces X_1 and X_2 also have this property. Pick points $x_1 \in X_1$, $x_2 \in X_2$ with $y := q_1(x_1) = q_2(x_2)$. According to the Lifting Theorem V.2.9, there exists a unique lift $\varphi\colon X_1 \to X_2$ of q_1 with $\varphi(x_1) = x_2$. We likewise obtain a unique lift $\psi\colon X_2 \to X_1$ of q_2 with $\psi(x_2) = x_1$. Then $\varphi \circ \psi\colon X_1 \to X_1$ is a lift of id_Y fixing x_1, so that the uniqueness of lifts implies that $\varphi \circ \psi = \mathrm{id}_{X_1}$. The same argument yields $\psi \circ \varphi = \mathrm{id}_{X_2}$, so that φ is a homeomorphism with the required properties. ■

Definition V.2.11. A topological space X is called *semilocally simply connected* if each point $x_0 \in X$ has a neighborhood U such that each loop $\alpha \in \Omega(U, x_0)$ is homotopic to $[x_0]$ in X, i.e., the natural homomorphism

$$\pi(i_U)\colon \pi_1(U, x_0) \to \pi_1(X, x_0), \quad [\gamma] \mapsto [i_U \circ \gamma]$$

induced by the inclusion map $i_U\colon U \to X$ is trivial. ■

Theorem V.2.12. *Let Y be arcwise connected and locally arcwise connected. Then Y has a simply connected covering space if and only if Y is semilocally simply connected.*

Proof. If $q\colon X \to Y$ is a simply connected covering space and $U \subseteq X$ is a pathwise connected elementary open subset, then each loop γ in U lifts to a loop $\widetilde{\gamma}$ in X, and since $\widetilde{\gamma}$ is homotopic to a constant map in X, the same holds for the loop $\gamma = q \circ \widetilde{\gamma}$ in Y.

Conversely, let us assume that Y is semilocally simply connected. We choose a base point $y_0 \in Y$ and let

$$\widetilde{Y} := P(Y, y_0)/\sim$$

be the set of homotopy classes of paths starting in y_0. We shall topologize $\widetilde{Y}$ in such a way that the map

$$q\colon \widetilde{Y} \to Y, \quad [\gamma] \mapsto \gamma(1)$$

defines a simply connected covering of Y.

Let $\mathcal{B}$ denote the set of all arcwise connected open subsets $U \subseteq Y$ for which each loop in U is contractible in Y and note that our assumptions on Y imply that $\mathcal{B}$ is a basis for the topology of Y, i.e., each open subset is a union of elements of $\mathcal{B}$. If $\gamma \in P(Y, y_0)$ satisfies $\gamma(1) \in U \in \mathcal{B}$, let

$$U_{[\gamma]} := \{[\eta] \in q^{-1}(U)\colon (\exists \beta \in C(I, U))\ \eta \sim \gamma * \beta\}.$$

We shall now verify several properties of these definitions, culminating in the proof of the theorem.

(1) $[\eta] \in U_{[\gamma]} \Rightarrow U_{[\eta]} = U_{[\gamma]}$.

To prove this, let $[\zeta] \in U_{[\eta]}$. Then $\zeta \sim \eta * \beta$ for some path β in U. Further $\eta \sim \gamma * \beta'$ for some path β' in U. Now $\zeta \sim \gamma * \beta' * \beta$, and $\beta' * \beta$ is a path in U, so that $[\zeta] \in U_{[\gamma]}$. This proves $U_{[\eta]} \subseteq U_{[\gamma]}$. We also have $\gamma \sim \eta * \overline{\beta'}$, so that $[\gamma] \in U_{[\eta]}$, and the first part implies that $U_{[\gamma]} \subseteq U_{[\eta]}$.

(2) q maps $U_{[\gamma]}$ injectively onto U.

That $q(U_{[\gamma]}) = U$ is clear since U and Y are arwise connected. To show that that it is one-to-one, let $[\eta], [\eta'] \in U_{[\gamma]}$, which we know from (1) is the same as $U_{[\eta]}$. Suppose $\eta(1) = \eta'(1)$. Since $[\eta'] \in U_{[\eta]}$, we have $\eta' \sim \eta * \alpha$ for some loop α in U. But then α is contractible in Y, so that $\eta' \sim \eta$, i.e., $[\eta'] = [\eta]$.

(3) $U, V \in \mathcal{B}$, $\gamma(1) \in U \subseteq V$, implies $U_{[\gamma]} \subseteq V_{[\gamma]}$.

This is trivial.

(4) The sets $U_{[\gamma]}$ for $U \in \mathcal{B}$ and $[\gamma] \in \widetilde{Y}$ form a basis of a topology on $\widetilde{Y}$.

Suppose $[\gamma] \in U_{[\eta]} \cap V_{[\eta']}$. Let $W \subseteq U \cap V$ be in $\mathcal{B}$ with $\gamma(1) \in W$. Then $[\gamma] \in W_{[\gamma]} \subseteq U_{[\gamma]} \cap V_{[\gamma]} = U_{[\eta]} \cap V_{[\eta']}$.

(5) q is open and continuous.

We have already seen in (2) that $q(U_{[\gamma]}) = U$, and these sets form a basis of the topology on $\widetilde{Y}$, resp., Y. Therefore q is an open map. We also have for $U \in \mathcal{B}$ the relation

$$q^{-1}(U) = \bigcup_{\gamma(1) \in U} U_{[\gamma]},$$

which is open. Hence q is continuous.

(6) $q|_{U_{[\gamma]}}$ is a homeomorphism.

This is because it is bijective, continuous and open.

At this point we have shown that $q: \widetilde{Y} \to Y$ is a covering map. It remains to see that $\widetilde{Y}$ is arcwise connected and simply connected.

(7) Let $H: I \times I \to Y$ be a continuous map with $H(t, 0) = y_0$. Then $h_t(s) := H(t, s)$ defines a path in Y starting in y_0. Let $\widetilde{h}(t) := [h_t] \in \widetilde{Y}$. Then $\widetilde{h}$ is a path in $\widetilde{Y}$ covering the path $t \mapsto h_t(1) = H(t, 1)$ in Y. We claim that $\widetilde{h}$ is continuous. Let $t_0 \in I$. We shall prove continuity at t_0. Let $U \in \mathcal{B}$ be a neighborhood of $h_{t_0}(1)$. Then there exists an interval $I_0 \subseteq I$ which is a neighborhood of t_0 with $h_t(1) \in U$ for $t \in I_0$. Then $\alpha(s) := H(t_0 + s(t - t_0), 1)$ is a continuous curve in U with $\alpha(0) = h_{t_0}(1)$ and $\alpha(1) = h_t(1)$, so that $h_{t_0} * \alpha$ is curve with the same endpoint as h_t. Applying Exercise V.2.1 to the restriction of H to the interval between t_0 and t, we see that $h_t \sim h_{t_0} * \alpha$, so that $\widetilde{h}(t) = [h_t] \in U_{[h_{t_0}]}$ for $t \in I_0$. Since $q|_{U_{[h_{t_0}]}}$ is a homeomorphism, $\widetilde{h}$ is continuous in t_0.

(8) $\widetilde{Y}$ is arcwise connected.

For $[\gamma] \in \widetilde{Y}$ put $h_t(s) := \gamma(st)$. By (7), this this yields a path $\widetilde{\gamma}(t) = [h_t]$ in $\widetilde{Y}$ from $\widetilde{y}_0 := [y_0]$ (the class of the constant path) to the point $[\gamma]$.

(9) $\widetilde{Y}$ is simply connected.

Let $\widetilde{\alpha} \in \Omega(\widetilde{Y}, \widetilde{y}_0)$ be a loop in $\widetilde{Y}$ and $\alpha := q \circ \widetilde{\alpha}$ its image in Y. Let $h_t(s) := \alpha(st)$. Then we have the path $\widetilde{h}(t) = [h_t]$ in $\widetilde{Y}$ from (7). This path covers α since $h_t(1) = \alpha(t)$. Further, $\widetilde{h}(0) = \widetilde{y}_0$ is the constant path. Also, by definition, $\widetilde{h}(1) = [\alpha]$. From the uniqueness of lifts we derive that $\widetilde{h} = \widetilde{\alpha}$ is closed, so that $[\alpha] = [y_0]$. Therefore the homomorphism

$$\pi_1(q): \pi_1(\widetilde{Y}, \widetilde{y}_0) \to \pi_1(Y, y_0)$$

vanishes. Since it is also injective (Corollary V.2.6), $\pi_1(\widetilde{Y}, \widetilde{y}_0)$ is trivial, i.e., $\widetilde{Y}$ is simply connected. ■

Definition V.2.13. Let $q: X \to Y$ be a covering. A homeomorphism $\varphi: X \to X$ is called a *deck tranformation* of the covering if $q \circ \varphi = \mathrm{id}_Y$. This means that φ permutes the elements in the fibers of q. We write $\mathrm{Deck}(X, q)$ for the group of deck transformations. ■

Example V.2.14. For the covering map $\exp: \mathbb{C} \to \mathbb{C}^\times$, the deck transformations have the form

$$\varphi(z) = z + 2\pi i n, \quad n \in \mathbb{Z}.$$ ■

Proposition V.2.15. *Let $q: \widetilde{Y} = P(Y, y_0)/\sim \to Y$ be the simply connected covering of Y with base point $\widetilde{y}_0 = [y_0]$. For each $[\gamma] \in \pi_1(Y, y_0)$ we write $\varphi_{[\gamma]} \in \mathrm{Deck}(\widetilde{Y}, q)$ for the unique lift of id_X mapping $[y_0]$ to the endpoint $[\gamma] = \widetilde{\gamma}(1)$ of the canonical lift $\widetilde{\gamma}$ of γ starting in $\widetilde{y}_0$. Then the map*

$$\Phi: \pi_1(Y, y_0) \to \mathrm{Deck}(\widetilde{Y}, q), \quad \Phi([\gamma]) = \varphi_{[\gamma]}$$

is an isomorphism of groups.

Proof. The composition $\varphi_{[\gamma]} \circ \varphi_{[\eta]}$ is a deck transformation mapping $\widetilde{y}_0$ to the endpoint of $\varphi_{[\gamma]} \circ \widetilde{\eta}$ which coincides with the endpoint of the lift of η starting in $\widetilde{\gamma}(1)$. Hence it also is the endpoint of the lift of the loop $\gamma * \eta$. Therefore Φ is a group homomorphism.

To see that Φ is injective, we note that $\varphi_\gamma = \mathrm{id}_{\widetilde{Y}}$ implies that $\widetilde{\gamma}(1) = \widetilde{y}_0$, so that $\widetilde{\gamma}$ is a loop, and hence that $[\gamma] = [y_0] = \widetilde{y}_0$.

For the surjectivity, let φ be a deck transformation and $y := \varphi(\widetilde{y}_0)$. If α is a path from $\widetilde{y}_0$ to y, then $\gamma := q \circ \alpha$ is a loop in y_0 with $\alpha = \widetilde{\gamma}$, so that $\varphi_\gamma(\widetilde{y}_0) = y$, and the uniqueness of lifts implies that $\varphi = \varphi_\gamma$. ■

Exercises for Section V.2.

Exercise V.2.1. Let $F: I^2 \to X$ be a continuous map with $F(0,s) = x_0$ for $s \in I$ and define

$$\gamma(t) := F(t,0), \quad \eta(t) := F(t,1), \quad \alpha(t) := F(1,t), \quad t \in I.$$

Show that $\gamma * \alpha \sim \eta$. Hint: Consider the map

$$G: I^2 \to I^2, \quad G(t,s) := \begin{cases} (2t, s) & \text{for } 0 \le t \le \frac{1}{2}, s \le 1 - 2t, \\ (1, 2t-1) & \text{for } \frac{1}{2} \le t \le 1, s \le 2t - 1, \\ (t + \frac{1-s}{2}, s) & \text{else} \end{cases}$$

and show that it is continuous. Take a look at the boundary values of $F \circ G$. ■

Exercise V.2.2. Let $q: G \to H$ be an morphism of topological groups with discrete kernel Γ.

(1) If $V \subseteq G$ is an open $\mathbf{1}$-neighborhood with $(V^{-1}V) \cap \Gamma = \{\mathbf{1}\}$ and q is open, then $q|_V: V \to q(V)$ is a homeomorphism.
(2) If q is open and surjective, then q is a covering.
(3) If q is open and H is connected, then q is surjective, hence a covering. ■

Exercise V.2.3. A map $f: X \to Y$ between topological spaces is called a *local homeomorphism* if each point $x \in X$ has an open neighborhood U such that $f|_U: U \to f(U)$ is a homeomorphism onto an open subset of Y.

(1) Show that each covering map is a local homeomorphism.
(2) Find a surjective local homeomorphism which is not a covering. Can you also find an example where X is connected? ■

VI. Applications to Lie groups

In this final chapter we eventually turn to applications of covering theory to Lie groups. Our coal is to see to which extent the Lie algebra and the fundamental group of a Lie group determine the group itself. To achieve this goal, we shall first show that each connected Lie group has a simply connected covering group $\widetilde{G}$ which also carries a Lie group structure and for which the kernel of the covering morphism $q_G: \widetilde{G} \to G$ can be identified with the fundamental group $\pi_1(G)$. Since $\mathbf{L}(q_G)$ is an isomorphism of Lie algebras, we then have $\mathbf{L}(G) \cong \mathbf{L}(\widetilde{G})$. In Section II we then prove the Monodromy Principle which leads to the result that any Lie algebra morphism $\mathbf{L}(G) \to \mathbf{L}(H)$ can be integrated to a group homomorphism, provided G is connected and simply connected. From that we shall derive that the Lie algebra $\mathbf{L}(G)$ determines the corresponding simply connected group up to isomorphy.

VI.1. Simply connected coverings of Lie groups

Theorem VI.1.1. *If G is a connected Lie group and $q_G: \widetilde{G} \to G$ its simply connected covering space. Then $\widetilde{G}$ carries a unique Lie group structure for which the covering map q_G is a smooth morphism of Lie groups.*

Proof. We first have to construct a (topological) group structure on the universal covering space $\widetilde{G}$. Pick an element $\widetilde{\mathbf{1}} \in q_G^{-1}(\mathbf{1})$. If $m_G: G \times G \to G$ denote the multiplication of G, then the map

$$m_G \circ (q_G \times q_G): \widetilde{G} \times \widetilde{G} \to G$$

lifts uniquely to a continuous map $\widetilde{m}_G: \widetilde{G} \times \widetilde{G} \to \widetilde{G}$ with $\widetilde{m}_G(\widetilde{\mathbf{1}}, \widetilde{\mathbf{1}}) = \widetilde{\mathbf{1}}$ (Theorem V.2.9). To see that the multiplication map $\widetilde{m}_G$ is associative, we observe that

$$\begin{aligned} &q_G \circ \widetilde{m}_G \circ (\mathrm{id}_{\widetilde{G}} \times \widetilde{m}_G) = m_G \circ (q_G \times q_G) \circ (\mathrm{id}_{\widetilde{G}} \times \widetilde{m}_G) \\ &= m_G \circ (\mathrm{id}_G \times m_G) \circ (q_G \times q_G \times q_G) = m_G \circ (m_G \times \mathrm{id}_G) \circ (q_G \times q_G \times q_G) \\ &= \cdots = q_G \circ \widetilde{m}_G \circ (\widetilde{m}_G \times \mathrm{id}_{\widetilde{G}}), \end{aligned}$$

so that the two continuous maps

$$\widetilde{m}_G \circ (\mathrm{id}_{\widetilde{G}} \times \widetilde{m}_G), \quad \widetilde{m}_G \circ (\widetilde{m}_G \times \mathrm{id}_{\widetilde{G}}): \widetilde{G}^3 \to \widetilde{G},$$

are lifts of the same map $\widetilde{G}^3 \to G$ and both map $(\widetilde{\mathbf{1}}, \widetilde{\mathbf{1}}, \widetilde{\mathbf{1}})$ to $\widetilde{\mathbf{1}}$. Hence the uniqueness of lifts implies that $\widetilde{m}_G$ is associative. We likewise obtain that the unique lift $\widetilde{\eta}_G: \widetilde{G} \to \widetilde{G}$ of the inversion map $\eta_G: G \to G$ with $\widetilde{\eta}_G(\widetilde{\mathbf{1}}) = \widetilde{\mathbf{1}}$ satisfies

$$\widetilde{m}_G \circ (\eta_{\widetilde{G}} \times \mathrm{id}_{\widetilde{G}}) = \widetilde{\mathbf{1}} = \widetilde{m}_G \circ (\mathrm{id}_{\widetilde{G}} \times \eta_{\widetilde{G}}).$$

Therefore $\widetilde{m}_G$ defines on $\widetilde{G}$ a topological group structure such that $q_G: \widetilde{G} \to G$ is a covering morphism of topological groups. Now Corollary IV.2.6 applies. ■

Proposition VI.1.2. *A surjective morphism $\varphi: G \to H$ of Lie groups is a covering if and only if $\mathbf{L}(\varphi): \mathbf{L}(G) \to \mathbf{L}(H)$ is a linear isomorphism.*

Proof. If φ is a covering, then it is an open momorphism with discrete kernel, so that $\mathbf{L}(\ker \varphi) = \{0\}$, and Theorem IV.1.16 implies that $\mathbf{L}(\varphi)$ is bijective, hence an isomorphism of Lie algebras.

If, conversely, $\mathbf{L}(\varphi)$ is bijective, then Theorem IV.1.16 implies that

$$\mathbf{L}(\ker \varphi) = \ker \mathbf{L}(\varphi) = \{0\},$$

and the Closed Subgroup Theorem shows that $\ker \varphi$ is discrete. Since $\mathbf{L}(\varphi)$ is an open map, the relation

$$\exp_H \circ \mathbf{L}(\varphi) = \varphi \circ \exp_G$$

entails that the image of any identity neighborhood in G is an identiy neighborhood in H. From

$$\varphi \circ \lambda_g = \lambda_{\varphi(g)} \circ \varphi$$

for each $g \in G$, we derive that this holds for any point in g, and therefore φ is an open map. Finally Exercise V.2.2 shows that φ is a covering. ■

Theorem VI.1.3. (Lifting Theorem for Groups) *Let $q: G \to H$ be a covering morphism of Lie groups. If $f: L \to H$ is a morphism of Lie groups, where L is connected and simply connected, then there exists a unique lift $\widetilde{f}: L \to G$ which is a morphism of Lie groups.*

Proof. Since Lie groups are locally arcwise connected, the Lifting Theorem V.2.9 implies the unique existence of a unique lift $\widetilde{f}$ with $\widetilde{f}(\mathbf{1}_L) = \mathbf{1}_G$. Then

$$m_G \circ (\widetilde{f} \times \widetilde{f}): L \times L \to G$$

is the unique lift of $m_H \circ (f \times f): L \times L \to H \times H$ mapping $(\mathbf{1}_L, \mathbf{1}_L)$ to $\mathbf{1}_G$. We also have

$$q \circ \widetilde{f} \circ m_L = f \circ m_L = m_H \circ (f \times f),$$

so that $\widetilde{f} \circ m_L$ is another lift mapping $(\mathbf{1}_L, \mathbf{1}_L)$ to $\mathbf{1}_G$. Therefore both are equal:

$$\widetilde{f} \circ m_L = m_G \circ (\widetilde{f} \times \widetilde{f}),$$

which means that $\widetilde{f}$ is a group homomorphism.

Since q is a local diffeomorphism and $\widetilde{f}$ is a continuous lift of f, it is also smooth in an identity neighborhood of L, hence smooth by Lemma IV.1.15. Alternatively, one can argue with the Automatic Smoothnes Theorem. ■

Lemma VI.1.4. *A discrete normal subgroup D of a connected topological group G is central.*

Proof. Let $d \in D$ and consider the map

$$\varphi\colon G \to D, \quad g \mapsto gdg^{-1}.$$

Then φ is continuous, and since D is discrete and G connected, φ is constant. We conclude that $gdg^{-1} = \varphi(g) = \varphi(\mathbf{1}) = d$ for each $g \in G$, i.e., that $d \in Z(G)$. ∎

Remark VI.1.5. If $q_G\colon \widetilde{G} \to G$ is the simply connected covering morphism of a connected Lie group G, then $\ker q_G$ is a discrete normal subgroup of the connected group $\widetilde{G}$, hence central by Lemma VI.1.4. Left multiplications by elements of $\ker q_G$ lead to deck transformations of the covering $\widetilde{G} \to G$, and this group of deck transformations acts transitively on the fiber $\ker q_G$ of $\mathbf{1}$. Proposition V.2.15 now shows that

$$\pi_1(G) \cong \ker q_G \tag{1.1}$$

as groups.

Since $q_G\colon \widetilde{G} \to G$ is open and surjective, we have $G \cong \widetilde{G}/\ker q_G$ as topological groups (Exercise VI.1.7). If, conversely, $\Gamma \subseteq \widetilde{G}$ is a discrete central subgroup, then the topological quotient group $\widetilde{G}/\Gamma$ is a Lie group (Corollary IV.2.6) whose universal covering group if $\widetilde{G}$. Two such groups $\widetilde{G}/\Gamma_1$ and $\widetilde{G}/\Gamma_2$ are isomorphic if and only if there exists a Lie group automorphism $\varphi \in \mathrm{Aut}(\widetilde{G})$ with $\varphi(\Gamma_1) = \Gamma_2$ (Exercise VI.1.8). Therefore the isomorphism classes of Lie groups with the same universal covering group $\widetilde{G}$ are parametrized by the orbits of the group $\mathrm{Aut}(\widetilde{G})$ in the set $\mathcal{S}$ of discrete central subgroups of $\widetilde{G}$. Since the normal subgroup $\mathrm{Inn}(\widetilde{G}) := \{c_g\colon g \in \widetilde{G}\}$ of inner automorphisms acts trivially on this set, the action of $\mathrm{Aut}(\widetilde{G})$ on $\mathcal{S}$ factors through an action of the group $\mathrm{Out}(\widetilde{G}) := \mathrm{Aut}(\widetilde{G})/\mathrm{Inn}(\widetilde{G})$.

Since each automorphism $\varphi \in \mathrm{Aut}(G)$ lifts to a unique automorphism $\widetilde{\varphi} \in \mathrm{Aut}(\widetilde{G})$ (Exercise VI.1.8), we have a natural embedding $\mathrm{Aut}(G) \hookrightarrow \mathrm{Aut}(\widetilde{G})$, and the image of this homomorphism consists of the stabilizer of the subgroup $\ker q_G \subseteq Z(\widetilde{G})$ in $\mathrm{Aut}(\widetilde{G})$. ∎

Example VI.1.6. Let A be a connected abelian Lie group and $\exp_A\colon \mathbf{L}(A) \to A$ its exponential function. Then $\exp_A$ is a morphism of Lie groups with $\mathbf{L}(\exp_A) = \mathrm{id}_{\mathbf{L}(A)}$, hence a covering morphism. Since $\mathbf{L}(A)$ is simply connected, we have $\mathbf{L}(A) \cong \widetilde{A}$ and $\ker \exp_A \cong \pi_1(A)$ is the fundamental group of A.

As special cases we obtain in particular the finite-dimensional tori

$$\mathbb{T}^d \cong \mathbb{R}^d/\mathbb{Z}^d \quad \text{with} \quad \pi_1(\mathbb{T}^n) \cong \mathbb{Z}^n.$$

It we want to classify all connected abelian Lie groups A of dimension n we can now proceed as follows. First we note that $\widetilde{A} \cong \mathbf{L}(A) \cong (\mathbb{R}^n, +)$ as abelian

Lie group. Then $\mathrm{Aut}(\widetilde{A}) \cong \mathrm{GL}_n(\mathbb{R})$ follows from the Automatic Smoothness Theorem IV.1.16. Further, Exercise III.3.1 implies that the discrete subgroup $\pi_1(A)$ of $\widetilde{A} \cong \mathbb{R}^n$ can be mapped by some $\varphi \in \mathrm{GL}_n(\mathbb{R})$ onto

$$\mathbb{Z}^k \cong \mathbb{Z}^k \times \{0\} \subseteq \mathbb{R}^k \times \mathbb{R}^{n-k} \cong \mathbb{R}^n.$$

Therefore

$$A \cong \mathbb{R}^n/\mathbb{Z}^k \cong \mathbb{T}^k \times \mathbb{R}^{n-k},$$

and it is clear that the number k is an isomorphy invariant of the Lie group A, it is the rank of its fundamental subgroup. Therefore connected abelian Lie groups A are determined up to isomorphism by the pair (n, k), where $n = \dim A$ and $k = \mathrm{rk}\,\pi_1(A)$. ■

As we have seen in Remark IV.1.5, it is important to have some information on the center of (simply) connected Lie groups. With the information in the following lemma, we are ready to deal with some important examples.

Lemma VI.1.7. *For a linear Lie group G, the kernel of the adjoint representation*

$$\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathbf{L}(G)), \quad \mathrm{Ad}(g).x = gxg^{-1}$$

is given by

$$Z_G(G_0) := \{g \in G \colon (\forall x \in G_0)\, gx = xg\}.$$

If, in addition, G is connected, then

$$\ker \mathrm{Ad} = Z(G).$$

Proof. (1) We recall that $\mathrm{Ad}(g)x = gxg^{-1}$. If $g \in Z_G(G_0)$, then for each $t \in \mathbb{R}$ we have

$$\exp(t\,\mathrm{Ad}(g).x) = g \exp txg^{-1} = \exp tx,$$

and therefore $\mathrm{Ad}(g).x = x$, so that $g \in \ker \mathrm{Ad}$. If, conversely, $g \in \ker \mathrm{Ad}$, then we obtain

$$\exp \mathbf{L}(G) \subseteq Z_G(g) = \{h \in G \colon hg = gh\}.$$

Now (1) leads to $G_0 \subseteq Z_G(g)$ which means that $g \in Z_G(G_0)$.

If, in addition, G is connected, then $G = G_0$ and $Z_G(G_0) = Z(G)$. ■

Below we shall use Lemma VI.1.7 to determine the kernel of the adjoint representation for various Lie groups. For that we have to know their center.

Examples VI.1.8. (a) First we recall from Proposition I.1.9 that $Z(\mathrm{GL}_n(\mathbb{K})) = \mathbb{K}^\times \mathbf{1}$. We claim that

$$Z(\mathrm{SL}_n(\mathbb{K})) = \{z\mathbf{1} \colon z \in \mathbb{K}^\times, z^n = 1\}.$$

From Lemma VI.1.7 we get

$$Z(\mathrm{SL}_n(\mathbb{K})) \subseteq \ker \mathrm{Ad} = \{g \in \mathrm{SL}_n(\mathbb{K}) \colon (\forall x \in \mathfrak{sl}_n(\mathbb{K})) gx = xg\}.$$

On the other hand, $\mathfrak{gl}_n(\mathbb{K}) = \mathbb{K}\mathbf{1} \oplus \mathfrak{sl}_n(\mathbb{K})$ shows that $\mathrm{Ad}(g) = \mathbf{1}$ implies that g commutes with all matrices, hence is central in $\mathrm{GL}_n(\mathbb{K})$. Therefore

$$Z(\mathrm{SL}_n(\mathbb{K})) = \mathrm{SL}_n(\mathbb{K}) \cap Z(\mathrm{GL}_n(\mathbb{K})) = \{z\mathbf{1}\colon z \in \mathbb{K}, z^n = 1\}.$$

In particular, we obtain

$$Z(\mathrm{SL}_n(\mathbb{C})) = \{z\mathbf{1}\colon z^n = 1\} \cong C_n$$

and

$$Z(\mathrm{SL}_n(\mathbb{R})) = \begin{cases} \mathbf{1} & \text{for } n \in 2\mathbb{N}_0 + 1 \\ \{\pm\mathbf{1}\} & \text{for } n \in 2\mathbb{N}_0. \end{cases}$$

(b) For $g \in Z(\mathrm{SU}_n(\mathbb{C})) = \ker \mathrm{Ad}$ we likewise have $gx = xg$ for all $x \in \mathfrak{su}_n(\mathbb{C})$. From

$$\mathfrak{gl}_n(\mathbb{C}) = \mathfrak{u}_n(\mathbb{C}) + i\mathfrak{u}_n(\mathbb{C}) = \mathfrak{su}_n(\mathbb{C}) + i\mathfrak{su}_n(\mathbb{C}) + \mathbb{C}\mathbf{1},$$

we derive that $g \in Z(\mathrm{GL}_n(\mathbb{C})) = \mathbb{C}^\times\mathbf{1}$. From that we immediately get

$$Z(\mathrm{SU}_n(\mathbb{C})) = \{z\mathbf{1}\colon z^n = 1\} \cong C_n$$

and similarly we obtain

$$Z(\mathrm{U}_n(\mathbb{C})) = \{z\mathbf{1}\colon |z| = 1\} \cong \mathbb{T}.$$

(c) Next we show that

$$Z(\mathrm{O}_n(\mathbb{R})) = \{\pm\mathbf{1}\} \quad \text{and} \quad Z(\mathrm{SO}_n(\mathbb{R})) = \begin{cases} \mathrm{SO}_2(\mathbb{R}) & \text{for } n = 2 \\ \mathbf{1} & \text{for } n \in 2\mathbb{N} + 1 \\ \{\pm\mathbf{1}\} & \text{for } n \in 2\mathbb{N} + 2. \end{cases}$$

If $g \in \mathrm{O}_n(\mathbb{R})$, then g commutes with each orthogonal reflection

$$\sigma_v\colon w \mapsto w - 2\langle v, w\rangle v$$

in the hyperplane $v^\perp$, where v is a unit vector. Since $\mathbb{R}v$ is the -1-eigenspace of σ_v, this space is invariant under g (Exercise I.1.1). This implies that for each $v \in \mathbb{R}^n$ we have $g.v \in \mathbb{R}v$ which by an elementary argument leads to $g \in \mathbb{R}^\times\mathbf{1}$ (Exercise VI.1.4). We conclude that

$$Z(\mathrm{O}_n(\mathbb{R})) = \mathrm{O}_n(\mathbb{R}) \cap \mathbb{R}^\times\mathbf{1} = \{\pm\mathbf{1}\}.$$

To determine the center of $\mathrm{SO}_n(\mathbb{R})$, we consider for orthogonal unit vectors v_1, v_2 the map $\sigma_{v_1,v_2} := \sigma_{v_1}\sigma_{v_2} \in \mathrm{SO}_n(\mathbb{R})$ (a reflection in the subspace $v_1^\top \cap v_2^\top$). Since an element $g \in Z(\mathrm{SO}_n(\mathbb{R}))$ commutes with σ_{v_1,v_2}, it leaves the plane $\mathbb{R}v_1 + \mathbb{R}v_2 = \ker(\sigma_{v_1,v_2} + \mathbf{1})$ invariant. If a linear map preserves all two-dimensional planes and $n \geq 3$, then it preserves all one-dimensional subspaces. As above, we get $g \in \mathbb{R}^\times\mathbf{1}$, which in turn leads to

$$Z(\mathrm{SO}_n(\mathbb{R})) = \mathrm{SO}_n(\mathbb{R}) \cap \mathbb{R}^\times\mathbf{1},$$

and the assertion follows. ∎

Example VI.1.9. We show that

$$\pi_1(\mathrm{SO}_3(\mathbb{R})) \cong C_2 = \{\pm 1\}$$

by constructing a surjective homomorphism

$$\varphi\colon \mathrm{SU}_2(\mathbb{C}) \to \mathrm{SO}_3(\mathbb{R})$$

with $\ker\varphi = \{\pm\mathbf{1}\}$. Since $\mathrm{SU}_2(\mathbb{C})$ is homeomorphic to $\mathbb{S}^3$, it is simply connected (Exercise V.1.3), so that we then obtain $\pi_1(\mathrm{SO}_3(\mathbb{R})) \cong C_2$ (Remark VI.1.5).

We consider

$$\mathfrak{su}_2(\mathbb{C}) = \{x \in \mathfrak{gl}_2(\mathbb{C})\colon x^* = -x, \operatorname{tr} x = 0\} = \Big\{ \begin{pmatrix} ai & b \\ -\bar{b} & -ai \end{pmatrix} : b \in \mathbb{C}, a \in \mathbb{R}\Big\}$$

and observe that this is a three-dimensional real subspace of $\mathfrak{gl}_2(\mathbb{C})$. We obtain on $E := \mathfrak{su}_2(\mathbb{C})$ the structure of a euclidean vector space by the scalar product

$$\beta(x,y) := \operatorname{tr}(xy^*) = -\operatorname{tr}(xy).$$

Now we consider the adjoint representation

$$\mathrm{Ad}\colon \mathrm{SU}_2(\mathbb{C}) \to \mathrm{GL}(E), \quad \mathrm{Ad}(g)(x) = gxg^{-1}.$$

Then we have for $x, y \in E$ and $g \in \mathrm{SU}_2(\mathbb{C})$ the relation

$$\begin{aligned}\beta\big(\mathrm{Ad}(g).x, \mathrm{Ad}(g).y\big) &= \operatorname{tr}(gxg^{-1}(gyg^{-1})^*) = \operatorname{tr}(gxg^{-1}(g^{-1})^*y^*g^*)\\ &= \operatorname{tr}(gxg^{-1}gy^*g^{-1}) = \operatorname{tr}(xy^*) = \beta(x,y).\end{aligned}$$

This means that

$$\mathrm{Ad}(\mathrm{SU}_2(\mathbb{C})) \subseteq \mathrm{O}(E,\beta) \cong \mathrm{O}_3(\mathbb{R}).$$

Since $\mathrm{SU}_2(\mathbb{C})$ is connected, we further obtain $\mathrm{Ad}(\mathrm{SU}_2(\mathbb{C})) \subseteq \mathrm{SO}(E,\beta) \cong \mathrm{SO}_3(\mathbb{R})$, the identity component of $\mathrm{O}(E,\beta)$.

The derived representation is given by

$$\mathrm{ad}\colon \mathfrak{su}_2(\mathbb{C}) \to \mathfrak{so}(E,\beta) \cong \mathfrak{so}_3(\mathbb{R}), \quad \mathrm{ad}(x)(y) = [x,y].$$

If $\operatorname{ad} x = 0$, then $\operatorname{ad} x(i\mathbf{1}) = 0$ implies that $\operatorname{ad} x(\mathfrak{u}_2(\mathbb{C})) = \{0\}$, so that $\operatorname{ad} x(\mathfrak{gl}_2(\mathbb{C})) = \{0\}$ follows from $\mathfrak{gl}_2(\mathbb{C}) = \mathfrak{u}_2(\mathbb{C}) + i\mathfrak{u}_2(\mathbb{C})$. This implies that $x \in \mathbb{C}\mathbf{1}$, so that $\operatorname{tr} x = 0$ leads to $x = 0$. Hence ad is injective, and we conclude with $\dim \mathfrak{so}(E,\beta) = \dim \mathfrak{so}_3(\mathbb{R}) = 3$ that

$$\mathrm{ad}(\mathfrak{su}_2(\mathbb{C})) = \mathfrak{so}(E,\beta)$$

Therefore

$$\operatorname{Im}\mathrm{Ad} = \langle \exp \mathfrak{so}(E,\beta)\rangle = \mathrm{SO}(E,\beta)_0 = \mathrm{SO}(E,\beta).$$

We thus obtain a surjective homomorphism

$$\varphi: \mathrm{SU}_2(\mathbb{C}) \to \mathrm{SO}_3(\mathbb{R}).$$

Since $\mathrm{SU}_2(\mathbb{C})$ is compact, the quotient group $\mathrm{SU}_2(\mathbb{C})/\ker\varphi$ is also compact, and the induced bijective morphism $\mathrm{SU}_2(\mathbb{C})/\ker\varphi \to \mathrm{SO}_3(\mathbb{R})$ is a homeomorphism (Exercise VI.1.7), hence an isomorphism of topological groups.

We further have

$$\ker\varphi = Z(\mathrm{SU}_2(\mathbb{C})) = \{\pm\mathbf{1}\}$$

(Exercise VI.1.7), so that

$$\widetilde{\mathrm{SO}}_3(\mathbb{R}) \cong \mathrm{SU}_2(\mathbb{C}) \quad \text{and} \quad \pi_1(\mathrm{SO}_3(\mathbb{R})) \cong C_2. \qquad \blacksquare$$

Examples VI.1.10. Without proof, we state some more results on fundamental groups of matrix groups (cf. Remark V.1.7 on polar decompositions).

(1) $\pi_1(\mathrm{SL}_n(\mathbb{C})) = \pi_1(\mathrm{SU}_n(\mathbb{C})) = \{\mathbf{1}\}$ because $\mathrm{SL}_n(\mathbb{C})$ has a polar decomposition.

In Example VI.1.8 we have seen that

$$Z(\mathrm{SL}_n(\mathbb{C})) = C_n\mathbf{1}$$

is a cyclic group. Therefore it is easy to determine all Lie groups whose universal covering group is $\mathrm{SL}_n(\mathbb{C})$. All these groups are of the form $\mathrm{SL}_n(\mathbb{C})/\Gamma$ for some discrete central subgroup. Since $Z(\mathrm{SL}_n(\mathbb{C}))$ is finite, it is discrete, and since it is cyclic, there exists for each divisor d of n a unique subgroup $\Gamma_d \subseteq Z(\mathrm{SL}_n(\mathbb{C}))$ with $|\Gamma_d| = d$. We conclude that the number of isomorphy classes of connected Lie groups G with $\widetilde{G} \cong \mathrm{SL}_n(\mathbb{C})$ coincides with the number of divisors of n.

(2) $\pi_1(\mathrm{GL}_n(\mathbb{C})) \cong \pi_1(\mathbb{C}^\times) \cong \mathbb{Z}$ because $\mathrm{GL}_n(\mathbb{C})$ has a polar decomposition and $\mathrm{GL}_n(\mathbb{C}) \cong \mathrm{SL}_n(\mathbb{C}) \times \mathbb{C}^\times$ (topologically).

Another way to this result is to observe that the multiplication map

$$\mu: \mathrm{SL}_n(\mathbb{C}) \times \mathbb{C}^\times \to \mathrm{GL}_n(\mathbb{C}), \quad (g,z) \mapsto zg$$

is a covering morphism of Lie groups. In fact, it is a smooth group homomorphism, and on the Lie algebra level

$$\mathbf{L}(\mu): \mathfrak{sl}_n(\mathbb{C}) \times \mathbb{C} \to \mathfrak{gl}_n(\mathbb{C}), \quad (x,y) \mapsto x + y\mathbf{1}$$

is a linear isomorphism (Proposition VI.1.2).

Since $\mathrm{SL}_n(\mathbb{C})$ is simply connected and $\widetilde{\mathbb{C}}^\times \cong \mathbb{C}$, we now see that

$$\widetilde{\mathrm{GL}}_n(\mathbb{C}) \cong \mathrm{SL}_n(\mathbb{C}) \times \mathbb{C} \quad \text{with} \quad Z(\widetilde{\mathrm{GL}}_n(\mathbb{C})) \cong C_n \times \mathbb{C}.$$

(3) Using the polar decomposition of $\mathrm{SL}_n(\mathbb{R})$, we get

$$\begin{aligned}\pi_1(\mathrm{SL}_n(\mathbb{R})) &= \pi_1(\mathrm{GL}_n(\mathbb{R})) = \pi_1(\mathrm{O}_n(\mathbb{R})) = \pi_1(\mathrm{SO}_n(\mathbb{R})) \\ &\cong \begin{cases} \mathbb{Z} & \text{for } n = 2 \\ \mathbb{Z}/2\mathbb{Z} & \text{for } n > 2 \end{cases}.\end{aligned}$$

Here the case $n = 2$ follows from $\mathrm{SO}_2(\mathbb{R}) \cong \mathbb{T}$ and the case $n = 3$ has been proved in Example VI.1.9. $\blacksquare$

Exercises for Section VI.1.

The following series of exercises leads to a classification of all (not necessarily connected) abelian linear Lie groups.

Exercise VI.1.1. An abelian group D is called *divisible* if for each $d \in D$ and $n \in \mathbb{N}$ there exists an $a \in D$ with $a^n = d$. Show that:

(1) If G is an abelian group, H a subgroup and $f: H \to D$ a homomorphism into a divisible group D, then there exists an extension of f to a homomorphism $\widetilde{f}: G \to D$. Hint: Use Zorn's Lemma to reduce the situation to the case where G is generated by H and one additional element.

(2) If G is an abelian group and D a divisible subgroup, then $G \cong D \times H$ for some subgroup H of G. Hint: Extend $\mathrm{id}_D: D \to D$ to a homomorphism $f: G \to D$ and define $H := \ker f$. ■

Exercise VI.1.2. Show that each abelian subgroup $A \subseteq \mathrm{GL}_n(\mathbb{C})$ is contained in a subgroup of the form $D \times U$, where $D \cong (\mathbb{C}^\times)^n$ is a subgroup consisting of diagonalizable matrizes and $U \cong \mathbb{C}^n$ is a group of unipotent matrices. Hint: Proceed in the following steps:

(1) If $g_1 = d_1 u_1$ and $g_2 = d_2 u_2$ are the multiplicative Jordan decompositions of $g_1, g_2 \in \mathrm{GL}_n(\mathbb{C})$ (Exercise II.2.5), then g_1 and g_2 commute if and only if $\{d_1, d_2, u_1, u_2\}$ is commutative.

(2) A_D denote the set of all diagonalizable Jordan components of the elements of A and A_U the set of all unipotent Jordan components. Show that A_D and A_U are abelian groups.

(3) A_D can be simultaneously diagonalized, hence is conjugate to a subgroup of the group $D \cong (\mathbb{C}^\times)^n$ of diagonal matrices in $\mathrm{GL}_n(\mathbb{C})$.

(4) $\log A_U$ is a commutative set of nilpotent matrices (Exercise III.1.4).

(5) $U := \exp(\mathrm{span}(\log A_U))$ is an abelian group of unipotent matrices isomorphic to the additive group of $\mathrm{span}(\log A_U)$ and commuting with A_D.

(6) $A \subseteq A_D \cdot U \cong A_D \times U$. ■

Exercise VI.1.3. (a) Let G be an abelian topological group for which G_0 is open and divisible. Show that $G \cong G_0 \times \pi_0(G)$, where $\pi_0(G) := G/G_0$ is considered as a discrete group.

(b) Show that each abelian Lie group is isomorphic to a group of the form $\mathbb{R}^n \times \mathbb{T}^m \times D$, where D is a discrete abelian group.

(c) If D is a discrete abelian linear group, then D is finitely generated. Hint: First one reduces with Exercise VI.1.2 to subgroups of $(\mathbb{C}^\times)^n \times \mathbb{C}^m$, and then further to subgroups of $\mathbb{C}^{n+m} \cong \mathbb{R}^{2(n+m)}$ (see also Exercise III.3.1).

(d) Show that each abelian linear Lie group is isomorphic to a group of the form $\mathbb{R}^n \times \mathbb{T}^m \times \mathbb{Z}^k \times E$, where E is a finite abelian group. Hint: Use (c) and the Structure Theorem for Finitely Generated Abelian Groups. ■

Exercise VI.1.4. Let V be a vector space over the field $\mathbb{K}$ and $A \in \mathrm{End}(V)$ with $Av \in \mathbb{K}v$ for all $v \in V$. Show that $A \in \mathbb{K}\,\mathrm{id}_V$. ■

Exercise VI.1.5. Let G be a connected linear Lie group. Show that

$$\mathbf{L}(Z(G)) = \mathfrak{z}(\mathbf{L}(G)) := \{x \in \mathbf{L}(G) \colon (\forall y \in \mathbf{L}(G))\ [x,y] = 0\}. \qquad ■$$

Exercise VI.1.6. Let X be a topological space and $(X_i)_{i\in I}$ connected subspaces of X with $X = \bigcup_{i\in I} X_i$. If $\bigcap_{i\in I} X_i \neq \emptyset$, then X is connected. ■

Exercise VI.1.7. If $q \colon G \to H$ is a surjective open morphism of topological groups, then the induced map $G/\ker q \to H$ is an isomorphism of topological groups, where $G/\ker q$ is endowed with the quotient topology. ■

Exercise VI.1.8. Let $q_G \colon \widetilde{G} \to G$ be a simply connected covering of the connected Lie group.

(1) Show that each automorphism $\varphi \in \mathrm{Aut}(G)$ has a unique lift $\widetilde{\varphi} \in \mathrm{Aut}(\widetilde{G})$.

(2) Derive from (1) that

$$\mathrm{Aut}(G) \cong \{\widetilde{\varphi} \in \mathrm{Aut}(\widetilde{G}) \colon \widetilde{\varphi}(\pi_1(G)) = \pi_1(G)\}.$$

(3) Show that, in general,

$$\{\widetilde{\varphi} \in \mathrm{Aut}(\widetilde{G}) \colon \widetilde{\varphi}(\pi_1(G)) \subseteq \pi_1(G)\}$$

is not a subgroup of $\mathrm{Aut}(\widetilde{G})$. Hint: Consider $q_G \colon \mathbb{R} \to \mathbb{T}, x \mapsto e^{2\pi i x}$. ■

Exercise VI.1.9. If G is a topological group and $\mathbf{1} \in U \subseteq G$ a connected subset. Then all sets $U^n := U \cdots U$ are connected and so is their union $\bigcup_n U^n$. ■

VI.2. More on homomorphisms of Lie groups

To round off the picture, we still have to provide the link between Lie algebras and covering groups. The main point is that in general one cannot integrate morphisms of Lie algebras $\mathbf{L}(G) \to \mathbf{L}(H)$ to morphisms of the corresponding groups $G \to H$, one has to assume that the group G is simply connected.

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Proposition VI.2.1. *Let G be a connected simply connected Lie group and H a group. Let V be an open symmetric connected identity neighborhood in G and $f \colon V \to H$ a function with*

$$f(xy) = f(x)f(y) \quad \textit{for} \quad x, y, xy \in V.$$

Then there exists a unique group homomorphism extending f. If, in addition, H is a Lie group and f is smooth, then its extension is also smooth.

Proof. We consider the group $G \times H$ and the subgroup $S \subseteq G \times H$ generated by the subset $U := \{(x, f(x)) : x \in V\}$. We endow U with the topology for which $x \mapsto (x, f(x)), V \to U$ is a homeomorphism. Note that $f(\mathbf{1})^2 = f(\mathbf{1}^2) = f(\mathbf{1})$ implies $f(\mathbf{1}) = \mathbf{1}$, which further leads to $\mathbf{1} = f(xx^{-1}) = f(x)f(x^{-1})$, so that $f(x^{-1}) = f(x)^{-1}$. Hence $U = U^{-1}$.

We now apply Lemma IV.2.3 because S is generated by U, and (T1/2) directly follow from the corresponding properties of V and $(x, f(x))(y, f(y)) = (xy, f(xy))$ for $x, y, xy \in V$. This leads to a group topology on S, for which S is a connected topological group. Indeed, its connectedness follows from $S = \bigcup_{n \in \mathbb{N}} U^n$ and the connectedness of all sets U^n (Exercise VI.1.6). The projection $p_G : G \times H \to G$ induces a covering homomorphism $q : S \to G$ because its restriction to the open $\mathbf{1}$-neighborhood is a homeomorphism (Exercise V.2.2(c)), and the connectedness of S and the simple connectedness of G imply that q is a homeomorphism (Corollary V.2.8). Now $F := p_H \circ q^{-1} : G \to H$ provides the required extension of f. In fact, for $x \in U$ we have $q^{-1}(x) = (x, f(x))$, and therefore $F(x) = f(x)$.

If, in addition, H is Lie and f is smooth, then the smoothness of the extension follows directly from Lemma IV.1.15. ■

Integrability Theorem for Lie Algebra Homomorphisms

Theorem VI.2.2. *Let G be a connected simply connected Lie group, H a Lie group and $\psi : \mathbf{L}(G) \to \mathbf{L}(H)$ a Lie algebra morphism. Then there exists a unique morphism $\varphi : G \to H$ with $\mathbf{L}(\varphi) = \psi$.*

Proof. Let $U \subseteq \mathbf{L}(G)$ be an open connected symmetric 0-neighborhood on which the BCH-product is defined and satisfies $\exp_G(x * y) = \exp_G(x)\exp_G(y)$ and $\exp_H(\psi(x) * \psi(y)) = \exp_H(\psi(x))\exp_H(\psi(y))$ for $x, y \in U$ (cf. Proposition III.2.6). Assume further that $\exp_G|_U$ is a homeomorphism onto an open subset of G (cf. Theorem IV.1.12).

The continuity of ψ and the fact that ψ is a Lie algebra homomorphism imply that for $x, y \in U$ the element $\psi(x * y)$ is the Dynkin-series for $\psi(x) * \psi(y)$. We define

$$f : \exp_G(U) \to H, \quad f(\exp_G(x)) := \exp_H(\psi(x)).$$

For $x, y, x * y \in U$ we then obtain

$$\begin{aligned} f(\exp_G(x)\exp_G(y)) &= f(\exp_G(x * y)) = \exp_H(\psi(x * y)) = \exp_H(\psi(x) * \psi(y)) \\ &= \exp_H(\psi(x))\exp_H(\psi(y)) = f(\exp_G(x))f(\exp_G(y)). \end{aligned}$$

Then $f : \exp(U) \to H$ satisfies the assumptions of Proposition VI.2.1, and we see that f extends uniquely to a group homomorphism $\varphi : G \to H$. Since $\exp_G$ is a local diffeomorphism, f is smooth in a $\mathbf{1}$-neighborhood, and therefore φ is smooth. We finally observe that φ is uniquely determined by $\mathbf{L}(\varphi) = \psi$ because G is connected (Theorem IV.1.16(1)). ■

The following corollary can be viewed as an integrability condition for ψ.

Corollary VI.2.3. *If G is a connected Lie group and H is a Lie group, then for a Lie algebra morphism $\psi\colon \mathbf{L}(G) \to \mathbf{L}(H)$ there exists a morphism $\varphi\colon G \to H$ with $\mathbf{L}(\varphi) = \psi$ if and only if $\pi_1(G) \subseteq \ker \widetilde{\varphi}$, where $\pi_1(G)$ is identified with the kernel of the universal covering map $q_G\colon \widetilde{G} \to G$ and $\widetilde{\varphi}\colon \widetilde{G} \to H$ is the unique morphism with $\mathbf{L}(\widetilde{\varphi}) = \psi \circ \mathbf{L}(q_G)$.*

Proof. If φ exists, then

$$(\varphi \circ q_G) \circ \exp_{\widetilde{G}} = \varphi \circ \exp_G \circ \mathbf{L}(q_G) = \exp_H \circ \psi \circ \mathbf{L}(q_G)$$

and the uniqueness of $\widetilde{\varphi}$ imply that $\widetilde{\varphi} = \varphi \circ q_G$ and hence that $\pi_1(G) = \ker q_G \subseteq \ker \widetilde{\varphi}$.

If, conversely, $\ker q_G \subseteq \ker \widetilde{\varphi}$, then $\varphi(q_G(g)) := \widetilde{\varphi}(g)$ defines a continuous morphism $G \cong \widetilde{G}/\ker q_G \to H$ with $\varphi \circ q_G = \widetilde{\varphi}$ (Exercise VI.1.7) and

$$\varphi \circ \exp_G \circ \mathbf{L}(q_G) = \varphi \circ q_G \circ \exp_{\widetilde{G}} = \widetilde{\varphi} \circ \exp_{\widetilde{G}} = \exp_H \circ \psi \circ \mathbf{L}(q_G). \qquad \blacksquare$$

Consequences for the classification of Lie groups

Let G and H be linear Lie groups. If $\varphi\colon G \to H$ is an isomorphism, then the functoriality of $\mathbf{L}$ directly implies that $\mathbf{L}(\varphi)\colon \mathbf{L}(G) \to \mathbf{L}(H)$ is an isomorphism. In fact, if $\psi\colon H \to G$ is a morphism with $\varphi \circ \psi = \mathrm{id}_H$ and $\psi \circ \varphi = \mathrm{id}_G$, then

$$\mathrm{id}_{\mathbf{L}(H)} = \mathbf{L}(\mathrm{id}_H) = \mathbf{L}(\varphi \circ \psi) = \mathbf{L}(\varphi) \circ \mathbf{L}(\psi)$$

and likewise $\mathbf{L}(\psi) \circ \mathbf{L}(\varphi) = \mathrm{id}_{\mathbf{L}(G)}$.

In this subsection we ask to which extent a Lie group G is determined by its Lie algebra $\mathbf{L}(G)$.

Theorem VI.2.4. *Two connected Lie groups G and H have isomorphic Lie algebras if and only if their universal covering groups $\widetilde{G}$ and $\widetilde{H}$ are isomorphic.*

Proof. If $\widetilde{G}$ and $\widetilde{H}$ are isomorphic, then we clearly have

$$\mathbf{L}(G) \cong \mathbf{L}(\widetilde{G}) \cong \mathbf{L}(\widetilde{H}) \cong \mathbf{L}(H).$$

Conversely, let $\psi\colon \mathbf{L}(G) \to \mathbf{L}(H)$ be an isomorphism. Using Proposition VI.2.2, we obtain a unique morphism $\varphi\colon \widetilde{G} \to \widetilde{H}$ with $\mathbf{L}(\varphi) = \psi$ and also a unique morphism $\widehat{\varphi}\colon \widetilde{H} \to \widetilde{G}$ with $\mathbf{L}(\widehat{\varphi}) = \psi^{-1}$. Then $\mathbf{L}(\varphi \circ \widehat{\varphi}) = \mathrm{id}_{\mathbf{L}(\widetilde{G})}$ implies $\varphi \circ \widehat{\varphi} = \mathrm{id}_{\widetilde{G}}$, and likewise $\widehat{\varphi} \circ \varphi = \mathrm{id}_{\widetilde{H}}$. Therefore $\widetilde{G}$ and $\widetilde{H}$ are isomorphic Lie groups. $\blacksquare$

Remark VI.2.5. Combining the preceding theorem with Remark VI.1.5, we see that if two connected Lie groups G and H have isomorphic Lie algebras, there exist discrete central subgroups $\Gamma_1, \Gamma_2 \subseteq \widetilde{G}$ with

$$G \cong \widetilde{G}/\Gamma_1 \quad \text{and} \quad H \cong \widetilde{G}/\Gamma_2.$$

We then have

$$\Gamma_1 \cong \pi_1(G) \quad \text{and} \quad \Gamma_2 \cong \pi_1(H).$$

In general, the condition $\pi_1(G) \cong \pi_1(H)$ does not suffice to conclude that G and H are isomorphic.

In fact, let

$$\widetilde{G} := \mathrm{SU}_2(\mathbb{C}) \times \mathrm{SU}_2(\mathbb{C})$$

whose center is $C_2 \times C_2$,

$$G := \widetilde{G}/(C_2 \times \{\mathbf{1}\}) \cong \mathrm{SO}_3(\mathbb{R}) \times \mathrm{SU}_2(\mathbb{C})$$

and

$$H := \widetilde{G}/\{(\mathbf{1},\mathbf{1}), (-\mathbf{1},-\mathbf{1})\} \cong \mathrm{SO}_4(\mathbb{R}),$$

where the latter isomorphy follows from Proposition VI.2.12 below. Then $\pi_1(G) \cong \pi_1(H) \cong C_2$, but there is no automorphism of $\widetilde{G}$ mapping $\pi_1(G)$ to $\pi_1(H)$.

Indeed, one can show that the two direct factors are the only non-trivial connected normal subgroups of $\widetilde{G}$, so that each automorphism of $\widetilde{G}$ either preserves both or exchanges them. Since $\pi_1(H)$ is not contained in any of them, it cannot be mapped to $\pi_1(G)$ by an automorphism of $\widetilde{G}$. ■

Examples VI.2.6. Here are some examples of pairs of linear Lie groups with isomorphic Lie algebras:

(1) $G = \mathrm{SO}_3(\mathbb{R})$ and $\widetilde{G} \cong \mathrm{SU}_2(\mathbb{C})$ (Example VI.1.9).

(2) $G = \mathrm{SO}_{2,1}(\mathbb{R})_0$ and $H = \mathrm{SL}_2(\mathbb{R})$: In this case we actually have a covering morphism $\varphi\colon H \to G$ coming from the adjoint representation

$$\mathrm{Ad}\colon \mathrm{SL}_2(\mathbb{R}) \to \mathrm{GL}(\mathbf{L}(H)) \cong \mathrm{GL}_3(\mathbb{R}).$$

On $\mathbf{L}(H) = \mathfrak{sl}_2(\mathbb{R})$ we consider the symmetric bilinear form given by $\beta(x,y) := \frac{1}{2}\operatorname{tr}(xy)$ and the basis

$$e_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_3 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then the matrix B of β with respect to this basis is

$$B := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

One easily verifies that

$$\operatorname{Im} \operatorname{Ad} \subseteq \mathrm{O}(\mathbf{L}(H), \beta) \cong \mathrm{O}_{2,1}(\mathbb{R}),$$

and since

$$\operatorname{ad}\colon \mathbf{L}(H) \to \mathfrak{o}_{2,1}(\mathbb{R})$$

is injective between spaces of the same dimension 3 (Exercise), it is bijective. Therefore

$$\operatorname{im} \operatorname{Ad} = \langle \exp \mathfrak{o}_{2,1}(\mathbb{R}) \rangle = \mathrm{SO}_{2,1}(\mathbb{R})_0$$

and Proposition VI.1.2 imply that

$$\operatorname{Ad}\colon \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SO}_{2,1}(\mathbb{R})_0$$

is a covering morphism. Its kernel is given by

$$Z(\mathrm{SL}_2(\mathbb{R})) = \{\pm \mathbf{1}\}.$$

From the polar decomposition one derives that both groups are homeomorphic to $\mathbb{T} \times \mathbb{R}^2$, and topologically the map Ad is like $(z, x, y) \mapsto (z^2, x, y)$, a two-fold covering.

(3) $G = \mathrm{SL}_2(\mathbb{C})$ and $H = \mathrm{SO}_{3,1}(\mathbb{R})_0$: Here we show that the universal covering group of the identity component H of the Lorentz group $\mathrm{SO}_{3,1}(\mathbb{R})$ is isomorphic to G. The construction follows a similar scheme as the argument in (2) above.

On the real 4-dimensional vector space $V := \mathrm{Herm}_2(\mathbb{C})$ we consider the representation

$$\sigma\colon G = \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}(V), \quad \sigma(g)(x) := gxg^*.$$

We want to find a symmetric bilinear form β on V invariant under the action of G and with $\mathrm{O}(V, \beta) \cong \mathrm{O}_{3,1}(\mathbb{R})$. We consider the symmetric bilinear form

$$\beta\colon V \times V \to \mathbb{R}, \quad \beta(x, y) := \operatorname{tr}(xy) - \operatorname{tr} x \operatorname{tr} y.$$

It is obvious that this form is symmetric. An orthogonal basis with respect to β is given by

$$e_1 := \mathbf{1}, \quad e_2 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_3 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_4 := \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},$$

and we have

$$\beta(e_1, e_1) = -2, \quad \beta(e_2, e_2) = \beta(e_3, e_3) = \beta(e_4, e_4) = 2.$$

Therefore $\mathrm{O}(V, \beta) \cong \mathrm{O}_{3,1}(\mathbb{R})$.

To see that $\operatorname{im}(\sigma) \subseteq \mathrm{O}(V,\beta)$, we observe that the quadratic form corresponding to β is

$$\beta(x,x) = \operatorname{tr} x^2 - (\operatorname{tr} x)^2 = -2\det x.$$

Now the invariance of β under G follows from the Polarization Identity and

$$\det(gxg^*) = \det g \det x \det g^* = \det x, \quad g \in \mathrm{SL}_2(\mathbb{C}), x \in \mathrm{Herm}_2(\mathbb{C}).$$

We conclude that $\sigma(G) \subseteq \mathrm{O}(V,\beta)$, and since G is connected, we further obtain $\sigma(G) \subseteq \mathrm{O}(V,\beta)_0 \cong \mathrm{SO}_{3,1}(\mathbb{R})_0$ (see also Exercise I.2.8). We also write σ for the corresponding homomorphism $G \to H = \mathrm{SO}_{3,1}(\mathbb{R})_0$.

The derived representation is given by

$$\mathbf{L}(\sigma)\colon \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{so}_{3,1}(\mathbb{R}), \quad \mathbf{L}(\sigma)(x)(y) = xy + yx^*.$$

If $\sigma(g) = \mathbf{1}$, then $gxg^* = x$ for all $x \in \mathrm{Herm}_2(\mathbb{C})$, and this implies that $g(ix)g^* = ix$, which leads with $M_2(\mathbb{C}) = \mathrm{Herm}_2(\mathbb{C}) + i\,\mathrm{Herm}_2(\mathbb{C})$ to $gxg^* = x$ for all $x \in M_2(\mathbb{C})$. For $x = g^*$ we obtain in particular $g^* = g^{-1}$. This in turn yields $gxg^{-1} = x$ for all $x \in M_2(\mathbb{C})$, so that $g \in \mathbb{C}^\times \mathbf{1}$, and thus $g \in \{\pm\mathbf{1}\}$. We conclude that $\ker\sigma = \{\pm\mathbf{1}\}$ is discrete and therefore $\ker \mathbf{L}(\sigma) \subseteq \mathbf{L}(\ker\sigma) = \{0\}$. Hence $\mathbf{L}(\sigma)$ is injective. Next $\dim \mathfrak{sl}_2(\mathbb{C}) = \dim \mathfrak{so}_{3,1}(\mathbb{R}) = 6$ shows that $\mathbf{L}(\sigma)$ is bijective, and we conclude that

$$\operatorname{im}(\sigma) = \langle \exp \operatorname{im} \mathbf{L}(\sigma)\rangle = \mathrm{SO}_{3,1}(\mathbb{R})_0 = H.$$

Therefore $\sigma\colon G \to H$ is a covering morphism (Proposition VI.1.2). In view of $\pi_1(\mathrm{SL}_2(\mathbb{C})) \cong \pi_1(\mathrm{SU}_2(\mathbb{C})) \cong \pi_1(\mathbb{S}^3) = \{\mathbf{1}\}$, it follows that

$$\mathrm{SL}_2(\mathbb{C}) \cong \widetilde{\mathrm{SO}}_{3,1}(\mathbb{R})_0. \qquad \blacksquare$$

Example VI.2.7. Let $G = \mathrm{SL}_2(\mathbb{R})$ and $H = \mathrm{SO}_{2,1}(\mathbb{R})_0$ and recall that $\widetilde{G} \cong \widetilde{H}$ follows from $\mathfrak{sl}_2(\mathbb{R}) \cong \mathfrak{so}_{2,1}(\mathbb{R})$ (cf. Example VI.2.6).

We further have $q_G(Z(\widetilde{G})) \subseteq Z(G) = \{\pm\mathbf{1}\}$ and $\pi_1(G) = \ker q_G \subseteq Z(\widetilde{G})$. Likewise

$$q_H(Z(\widetilde{G})) \subseteq Z(H) = \{\mathbf{1}\}$$

implies

$$Z(\widetilde{G}) \cong \pi_1(H) \cong \pi_1(\mathrm{O}_2(\mathbb{R}) \times \mathrm{O}_1(\mathbb{R})) \cong \mathbb{Z},$$

where the latter is a consequence of the polar decomposition. This implies that

$$Z(\widetilde{G}) \cong \mathbb{Z},$$

where

$$\pi_1(G) \cong 2\mathbb{Z} \quad \text{and} \quad \pi_1(H) \cong \mathbb{Z} = Z(\widetilde{G}).$$

Therefore G and H are not isomorphic, but they have isomorphic Lie algebras and isomorphic fundamental groups. $\blacksquare$

Non-linear Lie groups

We have already seen how to describe all connected Lie groups with a given Lie algebra. To determine all such groups which are, in addition, linear turns out to be a much more subtle enterprise. If $\widetilde{G}$ is a simply connected group with a given Lie algebra, it means to determine which of the groups $\widetilde{G}/D$ are linear. As the following examples show, the answer to this problem is not easy. In fact, a complete answer requires detailed knowledge of the structure of finite-dimensional Lie algebras.

Example VI.2.8. We show that the universal covering group $G := \widetilde{\mathrm{SL}_2}(\mathbb{R})$ of $\mathrm{SL}_2(\mathbb{R})$ is not a linear Lie group. Moreover, we show that every continuous homomorphism $\varphi\colon G \to \mathrm{GL}_n(\mathbb{R})$ satisfies $D := \pi_1(\mathrm{SL}_2(\mathbb{R})) \subseteq \ker\varphi$, hence factors through $G/D \cong \mathrm{SL}_2(\mathbb{R})$.

We consider the Lie algebra homomorphism $\mathbf{L}(\varphi)\colon \mathfrak{sl}_2(\mathbb{R}) \to \mathfrak{gl}_n(\mathbb{R})$. Then it is easy to see that

$$\mathbf{L}(\varphi)_{\mathbb{C}}(x+iy) := \mathbf{L}(\varphi).x + i\,\mathbf{L}(\varphi).y$$

defines an extension of $\mathbf{L}(\varphi)$ to a complex linear Lie algebra homomorphism

$$\mathbf{L}(\varphi)_{\mathbb{C}}\colon \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}_n(\mathbb{C}).$$

Since the group $\mathrm{SL}_2(\mathbb{C})$ is simply connected, there exists a unique group homomorphism $\psi\colon \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ with $\mathbf{L}(\psi) = \mathbf{L}(\varphi)_{\mathbb{C}}$.

Let $\alpha\colon G \to G/D \cong \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{C})$ be the canonical morphism. Then

$$\mathbf{L}(\varphi) = \mathbf{L}(\varphi)_{\mathbb{C}} \circ \mathbf{L}(\alpha) = \mathbf{L}(\psi) \circ \mathbf{L}(\alpha) = \mathbf{L}(\psi \circ \alpha)$$

implies $\varphi = \psi \circ \alpha$. We conclude that

$$\ker\varphi \supseteq \ker\alpha = D.$$

Therefore G has no faithful linear representation. ■

Lemma VI.2.9. *If A is a Banach algebra with unit $\mathbf{1}$ and $p, q \in A$ with $[p,q] = \lambda\mathbf{1}$, then $\lambda = 0$.*

Proof. By induction we obtain

$$[p, q^n] = \lambda n q^{n-1} \quad \text{for} \quad n \in \mathbb{N}. \tag{1.2}$$

In fact,

$$[p, q^{n+1}] = [p,q]q^n + q[p,q^n] = \lambda q^n + \lambda n q^n = \lambda(n+1)q^n.$$

Therefore

$$|\lambda|n\|q^{n-1}\| \leq 2\|p\|\|q^n\| \leq 2\|p\|\|q\|\|q^{n-1}\|$$

for each $n \in \mathbb{N}$, which leads to

$$(|\lambda|n - 2\|p\|\|q\|)\|q^{n-1}\| = 0.$$

If $\lambda \neq 0$, then we obtain for sufficiently large n that $q^{n-1} = 0$. For $n > 1$ we derive from (1.2) that $q^{n-2} = 0$. Inductively we arrive at the contradiction $q = 0$. ■

If A is a finite-dimensional algebra, we may w.l.o.g. assume that it is a subalgebra of some matrix algebra $M_n(\mathbb{K})$, and then $[p,q] = \lambda\mathbf{1}$ implies

$$n\lambda = \mathrm{tr}(\lambda\mathbf{1}) = \mathrm{tr}([p,q]) = 0$$

so that $\lambda = 0$.

Example VI.2.10. We consider the three-dimensional Heisenberg group

$$G = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

with the Lie algebra

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

Note that $\exp_G\colon \mathfrak{g} \to G$ is a diffeomorphism whose inverse is given by

$$\log(g) = (g - \mathbf{1}) - \frac{1}{2}(g - \mathbf{1})^2$$

(Proposition II.3.3). Let

$$z := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad p := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad q := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $[p,q] = z$, $[p,z] = [q,z] = 0$, $\exp \mathbb{R}z = \mathbf{1} + \mathbb{R}z \subseteq Z(G)$ and $D := \exp(\mathbb{Z}z)$ is a discrete central subgroup of G. We claim that the group G/D is not a linear Lie group. This will be verified by showing that each morphism $\alpha : G \to \mathrm{GL}_n(\mathbb{C})$ with $D \subseteq \ker\alpha$ satisfies $\exp(\mathbb{R}z) \subseteq \ker\alpha$.

The map $\mathbf{L}(\alpha)\colon \mathfrak{g} \to \mathfrak{gl}_n(\mathbb{C})$ is a Lie algebra homomorphism and we obtain linear maps

$$P := \mathbf{L}(\alpha)(p), \quad Q := \mathbf{L}(\alpha)(q) \quad \text{and} \quad Z := \mathbf{L}(\alpha)(z)$$

with $[P,Q] = Z$. Now $z \in D = \ker\alpha$ implies that $e^Z = \alpha(\exp z) = \mathbf{1}$ and hence that Z is diagonalizable with all eigenvalues contained in $2\pi i\mathbb{Z}$ (Exercise II.2.10). Let $V_\lambda := \ker(Z - \lambda\mathbf{1})$. Since z is central in $\mathfrak{g}$, the space V_λ is invariant under G (Exercise I.1.1), hence also under $\mathfrak{g}$ (Exercise III.2.5). Therefore the restrictions $P_\lambda := P\,|_{V_\lambda}$ and $Q_\lambda := Q\,|_{V_\lambda}$ satisfy $[P_\lambda, Q_\lambda] = \lambda\,\mathrm{id}$ in the Banach algebra $\mathrm{End}(V_\lambda)$. In view of the preceding lemma, we have $\lambda = 0$. Therefore the diagonalizability of Z entails that $Z = 0$ and hence that $\mathbb{R}z \subseteq \ker\alpha$. It follows in particular that the group G/D has no faithful linear representation. ■

The quaternions, $\mathrm{SU}_2(\mathbb{C})$ and $\mathrm{SO}_4(\mathbb{R})$

It is an important conceptual step to extend the real number field $\mathbb{R}$ to the field $\mathbb{C}$ of complex numbers. There are numerous motivations for this extension. The most obvious one is that not every algebraic equation with real coefficients has a solution in $\mathbb{R}$, and that $\mathbb{C}$ is *algebraically closed* in the sense that every non-constant polynomial, even with complex coefficients, has zeros in $\mathbb{C}$. This is the celebrated Fundamental Theorem of Algebra. For analysis the main point in passing from $\mathbb{R}$ to $\mathbb{C}$ is that the theory of holomorphic functions permits to understand many functions showing up in real analysis from a more natural viewpoint, which leads to a thorough understanding of singularities and of integrals which can be computed with the calculus of residues.

It therefore is a natural question whether there exists an extension $\mathbb{F}$ of the field $\mathbb{C}$ which would similarly enrich analysis and algebra if we pass from $\mathbb{C}$ to $\mathbb{F}$. It is an important algebraic result that there exists no finite-dimensional field extension of $\mathbb{R}$ other than $\mathbb{C}$. This is most naturally obtained in Galois theory, i.e., the theory of extending fields by adding zeros of polynomials. It is closely related to the fact that every real polynomial is a product of linear factors and factors of degree 2. Fortunately this does not mean that one has to give up, but that one has to sacrifice one of the axioms of a field to obtain something new.

We call a unital algebra A a *skew field* or a *division algebra* if every non-zero element $a \in A^\times$ is invertible, i.e., $A = A^\times \cup \{0\}$. Now the question is: Are there any division algebras which are finite-dimensional vector real spaces, apart from $\mathbb{R}$ and $\mathbb{C}$. Here the answer is yes: there is the four-dimensional division algebra $\mathbb{H}$ of *quaternions*, and this is the only finite-dimensional real non-commutative division algebra.

The easiest way to define the quaternions is to take

$$\mathbb{H} := \left\{ \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \in M_2(\mathbb{C}) \colon a, b \in \mathbb{C} \right\}.$$

Lemma VI.2.11. *$\mathbb{H}$ is a real subalgebra of $M_2(\mathbb{C})$ which is a division algebra.*

Proof. It is clear that $\mathbb{H}$ is a real vector subspace of $M_2(\mathbb{C})$. For the product of elements of $\mathbb{H}$ we obtain

$$\begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \begin{pmatrix} c & -\overline{d} \\ d & \overline{c} \end{pmatrix} = \begin{pmatrix} ac - \overline{b}d & -a\overline{d} - \overline{b}\overline{c} \\ bc + \overline{a}d & -b\overline{d} + \overline{ac} \end{pmatrix} \in \mathbb{H}.$$

This implies that $\mathbb{H}$ is a real subalgebra of $M_2(\mathbb{C})$.

We further have

$$\det \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} = |a|^2 + |b|^2, \tag{1.3}$$

so that every non-zero element of $\mathbb{H}$ is invertible in $M_2(\mathbb{C})$, and its inverse

$$\begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix}^{-1} = \frac{1}{|a|^2 + |b|^2} \begin{pmatrix} \overline{a} & \overline{b} \\ -b & a \end{pmatrix} \tag{1.4}$$

is again contained in $\mathbb{H}$. ■

A convenient basis for $\mathbb{H}$ is given by

$$\mathbf{1}, \quad I := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad K := IJ = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

Then the multiplication in $\mathbb{H}$ is completely determined by the relations

$$I^2 = J^2 = K^2 = -\mathbf{1} \quad \text{and} \quad IJ = -JI = K.$$

Here $\mathbb{C} \cong \mathbb{R}\mathbf{1} + \mathbb{R}I$, but $\mathbb{H}$ is not a complex algebra because the multiplication in $\mathbb{H}$ is not a complex bilinear map.

Since $\mathbb{H}$ is a division algebra, its group of units is $\mathbb{H}^\times = \mathbb{H} \setminus \{0\}$, and (1.3) implies that

$$\mathbb{H}^\times = \mathbb{H} \cap \mathrm{GL}_2(\mathbb{C}).$$

On $\mathbb{H}$ we consider the euclidean norm given by

$$|x| := \sqrt{\det x}, \quad \left| \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \right| = \sqrt{|a|^2 + |b|^2}.$$

From the multiplicativity of the determinant we immediately derive that

$$(1.5) \qquad |xy| = |x| \cdot |y| \quad \text{for} \quad x, y \in \mathbb{H}.$$

It follows in particular that $\mathbb{S} := \{x \in \mathbb{H} : |x| = 1\}$ is a subgroup of $\mathbb{H}$. In terms of complex matrices, we have $\mathbb{S} = \mathrm{SU}_2(\mathbb{C})$.

In this subsection we shall use the algebra structure of the quaternions to get some more information on the structure of the group $\mathrm{SO}_4(\mathbb{R})$. Here the idea is to identity $\mathbb{R}^4$ with $\mathbb{H}$.

Proposition VI.2.12. *We have a covering homomorphism*

$$\varphi: \mathrm{SU}_2(\mathbb{C}) \times \mathrm{SU}_2(\mathbb{C}) \to \mathrm{SO}_4(\mathbb{R}) \subseteq \mathrm{GL}(\mathbb{H}), \quad \varphi(a,b).x = axb^{-1}.$$

This homomorphism is a universal covering with $\ker \varphi = \{\pm(\mathbf{1}, \mathbf{1})\}$.

Proof. Since $|a| = |b| = 1$, formula (1.5) implies that all the maps $\varphi(a,b): \mathbb{H} \to \mathbb{H}$ are orthogonal, so that φ is a homomorphism

$$\mathrm{SU}_2(\mathbb{C}) \times \mathrm{SU}_2(\mathbb{C}) \to \mathrm{O}_4(\mathbb{R}).$$

Since $\mathrm{SU}_2(\mathbb{C}) \times \mathrm{SU}_2(\mathbb{C})$ is connected, it further follows that $\mathrm{im}(\varphi) \subseteq \mathrm{SO}_4(\mathbb{R})$.

To determine the kernel of φ, suppose that $\varphi(a,b) = \mathrm{id}_{\mathbb{H}}$. Then $axb^{-1} = x$ for all $x \in \mathbb{H}$. For $x = b$ we obtain in particular $a = b$. Hence $ax = xa$ for all $x \in \mathbb{H}$. With $x = I$ and $x = J$ this leads to $a \in \mathbb{R}\mathbf{1}$, and hence to $(a,b) \in \{\pm(\mathbf{1}, \mathbf{1})\}$. This proves the assertion on $\ker \varphi$.

The derived representation is given by

$$\mathbf{L}(\varphi): \mathfrak{su}_2(\mathbb{C}) \times \mathfrak{su}_2(\mathbb{C}) \to \mathfrak{so}_4(\mathbb{R}), \quad \mathbf{L}(\varphi)(x,y)(z) = xz - zy.$$

Since $\ker \varphi$ is discrete, it follows that $\ker \mathbf{L}(\varphi) \subseteq \mathbf{L}(\ker \varphi) = \{0\}$. Hence $\mathbf{L}(\varphi)$ is injective. Next $\dim \mathfrak{so}_4(\mathbb{R}) = 6 = 2 \dim \mathfrak{su}_2(\mathbb{C})$ shows that $\mathbf{L}(\varphi)$ is surjective, and we conclude that

$$\mathrm{im}(\varphi) = \langle \exp \mathrm{im}\, \mathbf{L}(\varphi) \rangle = \mathrm{SO}_4(\mathbb{R}).$$

Therefore φ is a covering morphism (Proposition VI.1.2). Since $\mathrm{SU}_2(\mathbb{C})$ is simply connected, $\widetilde{\mathrm{SO}_4}(\mathbb{R}) \cong \mathrm{SU}_2(\mathbb{C})^2$. ■

Let $G := \mathrm{SU}_2(\mathbb{C})^2$. We have just seen that this is the universal covering group of $\mathrm{SO}_4(\mathbb{R})$. On the other hand $\mathrm{SU}_2(\mathbb{C}) \cong \widetilde{\mathrm{SO}}_3(\mathbb{R})$. From $\mathbb{Z}(\mathrm{SU}_2(\mathbb{C})) = \{\pm\mathbf{1}\}$ we derive that

$$Z(G) = \{(\mathbf{1},\mathbf{1}),(\mathbf{1},-\mathbf{1}),(-\mathbf{1},\mathbf{1}),(-\mathbf{1},-\mathbf{1})\} \cong C_2^2.$$

We have

$$G/Z(G) \cong \mathrm{SO}_3(\mathbb{R}) \times \mathrm{SO}_3(\mathbb{R}),$$

and therefore

$$\mathrm{SO}_4(\mathbb{R})/\{\pm\mathbf{1}\} \cong G/Z(G) \cong \mathrm{SO}_3(\mathbb{R}) \times \mathrm{SO}_3(\mathbb{R}).$$

The group $\mathrm{SO}_4(\mathbb{R})$ is a twofold covering group of $\mathrm{SO}_3(\mathbb{R})^2$.

The End