III. Linear Lie groups

We call a closed subgroup $G \subseteq \mathrm{GL}_n(\mathbb{K})$ a *linear Lie group.* In this section we shall use the exponential function to assign to each linear Lie group G the vector space

$$\mathbf{L}(G) := \{x \in M_n(\mathbb{K}) \colon \exp(\mathbb{R}x) \subseteq G\}$$

called the *Lie algebra of* G. This subspace carries an additional algebraic structure because for $x, y \in \mathbf{L}(G)$ the commutator $[x,y] = xy - yx$ is contained in $\mathbf{L}(G)$, so that $[\cdot,\cdot]$ defines a skew-symmetric bilinear operation on $\mathbf{L}(G)$. As a first step, we shall see how to calculate $\mathbf{L}(G)$ for concrete groups and to use it to generalize the polar decomposition to a large class of linear Lie groups. In Section III.3 we shall then see how the Lie algebra $\mathbf{L}(G)$ determines the local structure of a the group G.

III.1. The Lie algebra of a linear Lie group

We start with the introduction of the concept of a Lie algebra.

Definition III.1.1. Let $\mathbb{K}$ be a field and L a $\mathbb{K}$-vector space. A bilinear map $[\cdot,\cdot]\colon L \times L \to L$ is called a *Lie bracket* if

(L1) $[x,x] = 0$ for $x \in L$ and

(L2) $\big[x,[y,z]\big] = \big[[x,y],z\big] + \big[y,[x,z]\big]$ for $x,y,z \in L$ (Jacobi identity[1]).

A *Lie algebra*[2] (over $\mathbb{K}$) is a $\mathbb{K}$-vector space L endowed with a Lie bracket. A subspace $E \subseteq L$ of a Lie algebra is called a *subalgebra* if $[E,E] \subseteq E$. A *homomorphism* $\varphi\colon L_1 \to L_2$ of Lie algebras is a linear map with $\varphi([x,y]) = [\varphi(x),\varphi(y)]$ for $x,y \in L_1$. A Lie algebra is said to be *abelian* if $[x,y] = 0$ holds for all $x,y \in L$. ■

The following lemma show that each associative algebra also carries a natural Lie algebra structure.

Lemma III.1.2. *Each associative algebra A is a Lie algebra A_L with respect to the commutator bracket*

$$[a,b] := ab - ba.$$

[1] Carl Gustav Jacob Jacobi (1804–1851), mathematician in Berlin and Königsberg (Kaliningrad). He found his famous identity about 1830 in the context of Poisson brackets, which are related to Hamiltonian mechanics and Symplectic Geometry.

[2] The notion of a Lie algebra was coined in the 1920s by Hermann Weyl.

Proof. (L1) is obvious. For (L2) we calculate

$$[a, bc] = abc - bca = (ab - ba)c + b(ac - ca) = [a, b]c + b[a, c],$$

and this implies

$$[a, [b, c]] = [a, b]c + b[a, c] - [a, c]b - c[a, b] = [[a, b], c] + [b, [a, c]]. \qquad \blacksquare$$

Definition III.1.3. A closed subgroup $G \subseteq \mathrm{GL}_n(\mathbb{K})$ is called a *linear Lie group*. For each subgroup $G \subseteq \mathrm{GL}_n(\mathbb{K})$ we define the set

$$\mathbf{L}(G) := \{x \in M_n(\mathbb{K}) \colon \exp(\mathbb{R}x) \subseteq G\}$$

and observe that $\mathbb{R}\,\mathbf{L}(G) \subseteq \mathbf{L}(G)$ follows immediately from the definition.

We could also define this notion in more abstract terms by considering a finite-dimensional $\mathbb{K}$-vector space V and call a closed subgroup $G \subseteq \mathrm{GL}(V)$ a linear Lie group. Then

$$\mathbf{L}(G) = \{x \in \mathrm{End}(V) \colon \exp(\mathbb{R}x) \subseteq G\}.$$

In the following we shall use both pictures. ∎

From Lemma III.1.2 we immediately derive that the associative algebra $M_n(\mathbb{K})$ is a Lie algebra with respect to the matrix commutator $[x, y] := xy - yx$.

The next proposition assigns a Lie algebra to each linear Lie group.

Proposition III.1.4. *If $G \subseteq \mathrm{GL}(V)$ is a closed subgroup, then $\mathbf{L}(G)$ is a real Lie subalgebra of $\mathrm{End}(V)_L$ and we obtain a map*

$$\exp_G \colon \mathbf{L}(G) \to G, \quad x \mapsto e^x.$$

In particular, we have

$$\mathbf{L}(\mathrm{GL}(V)) = \mathfrak{gl}(V) := \mathrm{End}(V)_L \quad \text{and} \quad \mathbf{L}(\mathrm{GL}_n(\mathbb{K})) = \mathfrak{gl}_n(\mathbb{K})) := M_n(\mathbb{K})_L.$$

We call $\mathbf{L}(G)$ the *Lie algebra of G*.

Proof. Let $x, y \in \mathbf{L}(G)$. For $k \in \mathbb{N}$ and $t \in \mathbb{R}$ we then have $\exp \frac{t}{k}x, \exp \frac{t}{k}y \in G$ and with the Trotter Formula (Proposition II.4.7), we get for all $t \in \mathbb{R}$:

$$\exp(t(x + y)) = \lim_{k\to\infty} \left(\exp \frac{tx}{k} \exp \frac{ty}{k}\right)^k \in G$$

because G is closed. Therefore $x + y \in \mathbf{L}(G)$.

Similarly we use the Commutator Formula to get

$$\exp t[x, y] = \lim_{k\to\infty} \left(\exp \frac{tx}{k} \exp \frac{y}{k} \exp -\frac{tx}{k} \exp -\frac{y}{k}\right)^{k^2} \in G,$$

hence $[x, y] \in \mathbf{L}(G)$. ∎

Lemma III.1.5. *Let $G \subseteq \mathrm{GL}_n(\mathbb{K})$ be a subgroup. If* $\mathrm{Hom}(\mathbb{R}, G)$ *denotes the set of all continuous group homomorphisms* $(\mathbb{R}, +) \to G$*, then the map*

$$\Gamma: \mathbf{L}(G) \to \mathrm{Hom}(\mathbb{R}, G), \quad x \mapsto \gamma_x, \quad \gamma_x(t) = \exp(tx)$$

is a bijection.

Proof. For each $x \in \mathbf{L}(G)$ the map $\gamma_x: \mathbb{R} \to G$ is a continuous group homomorphism (Theorem II.2.6), and since $x = \gamma_x'(0)$, the map Γ is injective. To see that it is surjective, let $\gamma: \mathbb{R} \to G$ be a continuous group homomorphism and $\iota: G \to \mathrm{GL}(V)$ the natural embedding. Then $\iota \circ \gamma: \mathbb{R} \to \mathrm{GL}(V)$ is a continuous group homomorphism, so that there exists an $x \in \mathfrak{gl}(V)$ with $\gamma(t) = \iota(\gamma(t)) = e^{tx}$ for all $t \in \mathbb{R}$ (Theorem II.2.6). Then $x \in \mathbf{L}(G)$ and $\gamma_x = \gamma$. ∎

Remark III.1.6. The preceding lemma implies in particular that for a linear Lie group G, the set $\mathbf{L}(G)$ can be defined directly in terms of the topological group structure on G as $\mathrm{Hom}(\mathbb{R}, G)$. This shows that $\mathbf{L}(G)$ does not depend on the special realization of G as a group of matrices. From the Trotter Formula and the Commutator Formula we also know that the Lie algebra structure on $\mathrm{Hom}(\mathbb{R}, G)$ can be defined intrinsically by

$$(\lambda\gamma)(t) := \gamma(\lambda t),$$

$$(\gamma_1 + \gamma_2)(t) := \lim_{n\to\infty} \Big(\gamma_1(\frac{t}{n})\gamma_2(\frac{t}{n})\Big)^{\frac{1}{n}}$$

and

$$[\gamma_1, \gamma_2](t) := \lim_{n\to\infty} \Big(\gamma_1(\frac{t}{n})\gamma_2(\frac{1}{n})\gamma_1(-\frac{t}{n})\gamma_2(-\frac{1}{n})\Big)^{\frac{1}{n^2}}.$$

Example III.1.7. We consider the homomorphism

$$\Phi: \mathbb{K}^n \to \mathrm{GL}_{n+1}(\mathbb{K}), \quad x \mapsto \begin{pmatrix} \mathbf{1} & x \\ 0 & 1 \end{pmatrix}$$

and observe that Φ is an isomorphism of the topological group $(\mathbb{K}^n, +)$ onto a linear Lie group.

The continuous one-parameter groups $\gamma: \mathbb{R} \to \mathbb{K}^n$ are easily determined because $\gamma(nt) = n\gamma(t)$ for all $n \in \mathbb{Z}$, $t \in \mathbb{R}$, implies further $\gamma(q) = q\gamma(1)$ for all $q \in \mathbb{Q}$ and hence, by continuity, $\gamma(t) = t\gamma(1)$ for all $t \in \mathbb{R}$. Since $(\mathbb{K}^n, +)$ is abelian, the Lie bracket on the Lie algebra $\mathbf{L}(\mathbb{K}^n, +)$ vanishes, and we obtain

$$\mathbf{L}(\mathbb{K}^n, +) = (\mathbb{K}^n, 0) \cong \mathbf{L}(\Phi(\mathbb{K}^n)) = \Big\{ \begin{pmatrix} \mathbf{0} & x \\ 0 & 0 \end{pmatrix} : x \in \mathbb{K}^n \Big\}$$

(Exercise). ∎

Functorial properties of the Lie algebra

So far we have assigned to each linear Lie group G its Lie algebra $\mathbf{L}(G)$. We shall also see that this assignment can be "extended" to continuous homomorphisms between linear Lie groups in the sense that we assign to each such homomorphism $\varphi: G_1 \to G_2$ a homomorphism $\mathbf{L}(\varphi): \mathbf{L}(G_1) \to \mathbf{L}(G_2)$ of Lie algebras, and this assignment satisfies

$$\mathbf{L}(\mathrm{id}_G) = \mathrm{id}_{\mathbf{L}(G)} \quad \text{and} \quad \mathbf{L}(\varphi_2 \circ \varphi_1) = \mathbf{L}(\varphi_2) \circ \mathbf{L}(\varphi_1)$$

for a composition $\varphi_1 \circ \varphi_2$ of two continuous homomorphisms $\varphi_1: G_2 \to G_1$ and $\varphi_2: G_2 \to G_3$. In the language of category theory, this means that $\mathbf{L}$ defines a functor from the category of linear Lie groups (where the morphisms are the continuous group homomorphisms) to the category of Lie algebras.

Proposition III.1.8. *Let $\varphi: G_1 \to G_2$ be a continuous group homomorphism of linear Lie groups. Then*

$$\mathbf{L}(\varphi)(x) := \frac{d}{dt}\Big|_{t=0} \varphi(\exp_{G_1}(tx))$$

defines a homomorphism of Lie algebras

$$\mathbf{L}(\varphi): \mathbf{L}(G_1) \to \mathbf{L}(G_2)$$

with

$$\exp_{G_2} \circ \mathbf{L}(\varphi) = \varphi \circ \exp_{G_1}, \tag{1.1}$$

i.e., the following diagram commutes

$$\begin{array}{ccc} G_1 & \xrightarrow{\varphi} & G_2 \\ \uparrow{\scriptstyle \exp_{G_1}} & & \uparrow{\scriptstyle \exp_{G_2}} \\ \mathbf{L}(G_1) & \xrightarrow{\mathbf{L}(\varphi)} & \mathbf{L}(G_2). \end{array}$$

Conversely, $\mathbf{L}(\varphi)$ is the uniquely determined linear map satisfying (1.1).

Proof. For $x \in \mathbf{L}(G_1)$ we consider the homomorphism $\gamma_x \in \mathrm{Hom}(\mathbb{R}, G_1)$ given by $\gamma_x(t) = e^{tx}$. According to Lemma III.1.5, we have

$$\varphi \circ \gamma_x(t) = \exp_{G_2}(ty)$$

for some $y \in \mathbf{L}(G_2)$, because $\varphi \circ \gamma_x: \mathbb{R} \to G_2$ is a continuous group homomorphism. Then clearly $y = (\varphi \circ \gamma_x)'(0) = \mathbf{L}(\varphi)x$. For $t = 1$ we obtain in particular

$$\exp_{G_2}(\mathbf{L}(\varphi)x) = \varphi(\exp_{G_1}(x),$$

which is (1.1).

Conversely, every linear map $\psi\colon \mathbf{L}(G_1) \to \mathbf{L}(G_2)$ with

$$\exp_{G_2} \circ \psi = \varphi \circ \exp_{G_1}$$

satisfies

$$\varphi \circ \exp_{G_1}(tx) = \exp_{G_2}(\psi(tx)) = \exp_{G_2}(t\psi(x)),$$

and therefore

$$\mathbf{L}(\varphi)x = \frac{d}{dt}\Big|_{t=0} \exp_{G_2}(t\psi(x)) = \psi(x).$$

Next we show that $\mathbf{L}(\varphi)$ is a homomorphism of Lie algebras. Let $x \in \mathbf{L}(G_1)$. From the definition of $\mathbf{L}(\varphi)$ we immediately get

$$\exp(s\,\mathbf{L}(\varphi)(tx)) = \varphi(\exp(stx)) = \exp(ts\,\mathbf{L}(\varphi)(x)), \quad s,t \in \mathbb{R},$$

which leads to $\mathbf{L}(\varphi)(tx) = t\,\mathbf{L}(\varphi)(x)$.

Since φ is continuous, the Trotter Formula implies that

$$\begin{aligned}
\exp(\mathbf{L}(\varphi)(x+y)) &= \varphi\big(\exp(x+y)\big)\\
&= \lim_{k\to\infty} \varphi\Big(\exp\frac{1}{k}x \exp\frac{1}{k}y\Big)^k\\
&= \lim_{k\to\infty} \Big(\varphi\Big(\exp\frac{1}{k}x\Big)\varphi\Big(\exp\frac{1}{k}y\Big)\Big)^k\\
&= \lim_{k\to\infty} \Big(\exp\frac{1}{k}\,\mathbf{L}(\varphi)(x)\exp\frac{1}{k}\,\mathbf{L}(\varphi)(y)\Big)^k\\
&= \exp\big(\mathbf{L}(\varphi)(x) + \mathbf{L}(\varphi)(y)\big)
\end{aligned}$$

for all $x,y \in \mathbf{L}(G_1)$. Therefore $\mathbf{L}(\varphi)(x+y) = \mathbf{L}(\varphi)(x) + \mathbf{L}(\varphi)(y)$ because the same formula holds with tx and ty instead of x and y. Hence $\mathbf{L}(\varphi)$ is additive and therefore linear.

We likewise obtain with the Commutator Formula

$$\varphi(\exp[x,y]) = \exp[\mathbf{L}(\varphi)(x), \mathbf{L}(\varphi)(y)]$$

and thus

$$\mathbf{L}(\varphi)([x,y]) = [\mathbf{L}(\varphi)(x), \mathbf{L}(\varphi)(y)].$$ ■

Corollary III.1.9. *If $\varphi_1 : G_1 \to G_2$ and $\varphi_2\colon G_2 \to G_3$ are continuous homomorphisms of linear Lie groups, then*

$$\mathbf{L}(\varphi_2 \circ \varphi_1) = \mathbf{L}(\varphi_2) \circ \mathbf{L}(\varphi_1).$$

Moreover,

$$\mathbf{L}(\mathrm{id}_G) = \mathrm{id}_{\mathbf{L}(G)}\,.$$

Proof. We have the relations

$$\varphi_1 \circ \exp_{G_1} = \exp_{G_2} \circ \mathbf{L}(\varphi_1) \quad \text{and} \quad \varphi_2 \circ \exp_{G_2} = \exp_{G_3} \circ \mathbf{L}(\varphi_2),$$

which immediately lead to

$$(\varphi_2 \circ \varphi_1) \circ \exp_{G_1} = \varphi_2 \circ \exp_{G_2} \circ \mathbf{L}(\varphi_1) = \exp_{G_3} \circ (\mathbf{L}(\varphi_2) \circ \mathbf{L}(\varphi_1)),$$

and the uniqueness assertion of Proposition III.1.8 implies that

$$\mathbf{L}(\varphi_2 \circ \varphi_1) = \mathbf{L}(\varphi_2) \circ \mathbf{L}(\varphi_1).$$

Clearly $\mathrm{id}_{\mathbf{L}(G)}$ is a linear map satisfying

$$\exp_G \circ \mathrm{id}_{\mathbf{L}(G)} = \mathrm{id}_G \circ \exp_G,$$

so that the uniqueness assertion of Proposition III.1.8 implies $\mathbf{L}(\mathrm{id}_G) = \mathrm{id}_{\mathbf{L}(G)}$. ∎

Corollary III.1.10. *If $\varphi\colon G_1 \to G_2$ is an isomorphism of linear Lie groups, then $\mathbf{L}(\varphi)$ is an isomorphism of Lie algebras.*

Proof. Since φ is an isomorphism of linear Lie groups, it is bijective and $\psi := \varphi^{-1}$ also is a continuous homomorphism. We then obtain with Corollar III.1.9 the relations

$$\mathrm{id}_{\mathbf{L}(G_2)} = \mathbf{L}(\mathrm{id}_{G_2}) = \mathbf{L}(\varphi \circ \psi) = \mathbf{L}(\varphi) \circ \mathbf{L}(\psi)$$

and likewise

$$\mathrm{id}_{\mathbf{L}(G_1)} = \mathbf{L}(\psi) \circ \mathbf{L}(\varphi).$$

Hence $\mathbf{L}(\varphi)$ is an isomorphism with $\mathbf{L}(\varphi)^{-1} = \mathbf{L}(\psi)$. ∎

Definition III.1.11. If V is a vector space and G a group, then a homomorphism $\varphi\colon G \to \mathrm{GL}(V)$ is called a *representation of G on V*. If $\mathfrak{g}$ is a Lie algebra, then a homomorphism of Lie algebras $\varphi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is called a *representation of $\mathfrak{g}$ on V*. ∎

As a consequence of Proposition III.1.8, we obtain

Corollary III.1.12. *If $\varphi\colon G \to \mathrm{GL}(V)$ is a continuous representation of the linear Lie group G, then $\mathbf{L}(\varphi)\colon \mathbf{L}(G) \to \mathfrak{gl}(V)$ is a representation of the Lie algebra $\mathbf{L}(G)$.* ∎

The representation $\mathbf{L}(\varphi)$ obtained in Corollary III.1.12 from the group representation φ is called the *derived representation.* This is motivated by the fact that for each $x \in \mathbf{L}(G)$ we have

$$\mathbf{L}(\varphi)(x) = \frac{d}{dt}\Big|_{t=0} e^{t\,\mathbf{L}(\varphi)(x)} = \frac{d}{dt}\Big|_{t=0} \varphi(\exp tx).$$

The adjoint representation

Let $G \subseteq \mathrm{GL}(V)$ be a linear Lie group and $\mathbf{L}(G) \subseteq \mathfrak{gl}(V)$ the corresponding Lie algebra. For $g \in G$ we define the conjugation automorphism $c_g \in \mathrm{Aut}(G)$ by $c_g(x) := gxg^{-1}$. Then

$$\begin{aligned}\mathbf{L}(c_g)(x) &= \frac{d}{dt}\Big|_{t=0} c_g(\exp tx) = \frac{d}{dt}\Big|_{t=0} g(\exp tx)g^{-1} \\ &= \frac{d}{dt}\Big|_{t=0} \exp(tgxg^{-1}) = gxg^{-1}\end{aligned}$$

(Lemma II.2.2), and therefore $\mathbf{L}(c_g) = c_g|_{\mathbf{L}(G)}$. We define the *adjoint representation of G on* $\mathbf{L}(G)$ by

$$\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathbf{L}(G)), \quad \mathrm{Ad}(g)(x) := \mathbf{L}(c_g)x = gxg^{-1}.$$

(That this is a representation follows immediately from the explicit formula).

For each $x \in \mathbf{L}(G)$ the map $G \to \mathbf{L}(G), g \mapsto \mathrm{Ad}(g)(x) = gxg^{-1}$ is continuous and each $\mathrm{Ad}(g)$ is an automorphism of the Lie algebra $\mathbf{L}(G)$. Therefore Ad is a continuous homomorphism from the linear Lie group G to the linear Lie group $\mathrm{Aut}(\mathbf{L}(G)) \subseteq \mathrm{GL}(\mathbf{L}(G))$. The derived representation

$$\mathbf{L}(\mathrm{Ad})\colon \mathbf{L}(G) \to \mathfrak{gl}(\mathbf{L}(G))$$

is a representation of $\mathbf{L}(G)$ on $\mathbf{L}(G)$. The following lemma gives a formula for this representation.

For $x \in \mathbf{L}(G)$ we define $\mathrm{ad}(x)\colon \mathbf{L}(G) \to \mathbf{L}(G)$ by $\mathrm{ad}\, x(y) := [x, y]$.

Lemma III.1.13. $\mathbf{L}(\mathrm{Ad}) = \mathrm{ad}$.

Proof. This is an immediate consequence of the relation

$$\mathrm{Ad}(\exp x) = e^{\mathrm{ad}\, x} \tag{1.2}$$

(Lemma II.4.1). ■

Exercises for Section III.1.

Exercise III.1.1. (a) If $(G_j)_{j\in J}$ is a family of linear Lie groups in $\mathrm{GL}_n(\mathbb{R})$, then their intersection $G := \bigcap_{j\in J} G_j$ also is a linear Lie group.

(b) If $(G_j)_{j\in J}$ is a family of subgroups of $\mathrm{GL}_n(\mathbb{K})$, then

$$\mathbf{L}\Big(\bigcap_{j\in J} G_j\Big) = \bigcap_{j\in J} \mathbf{L}(G_j).$$ ■

Exercise III.1.2. Let $G := \mathrm{GL}_n(\mathbb{K})$ and $V := P_k(\mathbb{K}^n)$ the space of homogeneous polynomials of degree k in $x_1, \ldots, x_n$, considered as functions $\mathbb{K}^n \to \mathbb{K}$. Show that:

(1) $\dim V = \binom{k+n-1}{n-1}$.

(2) We obtain a continuous representation $\rho: G \to \mathrm{GL}(V)$ of G on V by $(\rho(g).f)(x) := f(g^{-1}x)$.

(3) For the elementary matrix $E_{ij} = (\delta_{ij})$ we have $\mathbf{L}(\rho)(E_{ij}) = -x_j \frac{\partial}{\partial x_i}$. Hint: $(\mathbf{1} + tE_{ij})^{-1} = \mathbf{1} - tE_{ij}$. ■

Exercise III.1.3. (a) If $X \in \mathrm{End}(V)$ is nilpotent, then $\mathrm{ad}\, X \in \mathrm{End}(\mathrm{End}(V))$ is also nilpotent. Hint: $\mathrm{ad}\, X = L_X - R_X$ and both summands commute.

(b) If $X \in \mathrm{End}(V)$ is diagonalizable with the eigenvalues $\lambda_1, \ldots, \lambda_n$, then $\mathrm{ad}\, X \in \mathrm{End}(\mathrm{End}(V))$ is also diagonalizable with the eigenvalues $\lambda_i - \lambda_j$, $i, j = 1, \ldots, n$.

(c) If $X = X_s + X_n$ is the additive Jordan decomposition of X, then $\mathrm{ad}\, X = \mathrm{ad}(X_s) + \mathrm{ad}(X_n)$ is the additive Jordan decomposition of $\mathrm{ad}\, X$. ■

Exercise III.1.4. If $X, Y \in M_n(\mathbb{K})$ are nilpotent, then the following are equivalent:

(1) $\exp X \exp Y = \exp Y \exp X$.

(2) $[X, Y] = 0$.

Hint: (1) implies $\exp X \exp Y \exp -X = \exp(e^{\mathrm{ad}\, X}.Y) = \exp Y$. Now conclude that $e^{\mathrm{ad}\, X}.Y = Y$ (Proposition II.3.3) and then use Exercise III.1.3 and Corollary II.3.4. ■

III.2. Calculating Lie algebras of linear Lie groups

In this section we shall see various techniques to determine the Lie algebra of a linear Lie group.

Example III.2.1. The group $G := \mathrm{SL}_n(\mathbb{K}) = \det^{-1}(1) = \ker \det$ is a linear Lie group. To determine its Lie algebra, we first claim that

$$\det(e^x) = e^{\mathrm{tr}\, x} \tag{2.1}$$

holds for $x \in M_n(\mathbb{K})$. To verify this claim, we consider

$$\det: M_n(\mathbb{K}) \cong (\mathbb{K}^n)^n \to \mathbb{K}$$

as a multilinear map, where each matrix x is considered as an n-tuple of its column vectors $x_1, \ldots, x_n$. Then Lemma A.2.6 on the differential of a multilinear map implies that

$$\begin{aligned}(\mathtt{d}\det)(\mathbf{1})(x) &= (\mathtt{d}\det)(e_1, \ldots, e_n)(x_1, \ldots, x_n)\\ &= \det(x_1, e_2, \ldots, e_n) + \ldots + \det(e_1, \ldots, e_{n-1}, x_n)\\ &= x_{11} + \ldots + x_{nn} = \mathrm{tr}\, x.\end{aligned}$$

Now we consider the curve $\gamma\colon \mathbb{R} \to \mathbb{K}^\times \cong \mathrm{GL}_1(\mathbb{K}), t \mapsto \det(e^{tx})$. Then γ is a continuous group homomorphism, hence of the form $\gamma(t) = e^{at}$ for $a = \gamma'(0)$ (Theorem II.2.6). On the other hand, the Chain Rule implies

$$a = \gamma'(0) = \mathtt{d}\det(\mathbf{1})\big(\mathtt{d}\exp(\mathbf{0})(x)\big) = \mathrm{tr}(x),$$

and this implies (2.1).

We conclude that

$$\begin{aligned}\mathfrak{sl}_n(\mathbb{K}) &:= \mathbf{L}(\mathrm{SL}_n(\mathbb{K})) = \{x \in M_n(\mathbb{K})\colon (\forall t \in \mathbb{R})\, 1 = \det(e^{tx}) = e^{t\,\mathrm{tr}\,x}\}\\ &= \{x \in M_n(\mathbb{K})\colon \mathrm{tr}\,x = 0\}.\end{aligned}$$

■

Lemma III.2.2. *Let V and W be finite-dimensional vector spaces and $\beta\colon V \times V \to W$ a bilinear map. For $(x, y) \in \mathrm{End}(V) \times \mathrm{End}(W)$ the following are equivalent:*

(1) $e^{ty}.\beta(v, v') = \beta(e^{tx}.v, e^{tx}.v')$ *for all* $t \in \mathbb{R}$ *and all* $v, v' \in V$.

(2) $y.\beta(v, v') = \beta(x.v, v') + \beta(v, x.v')$ *for all* $v, v' \in V$.

Proof. (1) $\Rightarrow$ (2): Taking the derivative in $t = 0$, the relation (1) leads to

$$y.\beta(v, v') = \beta(x.v, v') + \beta(v, x.v'),$$

where we use the Product and the Chain Rule (cf. Theorem A.2.3, Lemma A.2.6).

(2) $\Rightarrow$ (1): If (2) holds, then we obtain inductively

$$y^n.\beta(v, v') = \sum_{k=0}^{n} \binom{n}{k} \beta(x^k.v, x^{n-k}.v').$$

For the exponential series this leads with the general Cauchy Product Formula (Exercise II.1.3) to

$$\begin{aligned} e^y.\beta(v, v') &= \sum_{n=0}^{\infty} \frac{1}{n!} y^n.\beta(v, v') = \sum_{n=0}^{\infty} \frac{1}{n!} \Big(\sum_{k=0}^{n} \binom{n}{k} \beta(x^k.v, x^{n-k}.v) \Big) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \beta\left(\frac{1}{k!} x^k.v, \frac{1}{(n-k)!} x^{n-k}.v' \right) \\ &= \beta\left(\sum_{k=0}^{\infty} \frac{1}{k!} x^k.v, \sum_{m=0}^{\infty} \frac{1}{m!} x^m.v' \right) \\ &= \beta\left(e^x.v, e^x.v'\right). \end{aligned}$$

Since (2) also holds for the pair (tx, ty) for all $t \in \mathbb{R}$, this completes the proof.■

Proposition III.2.3. *Let V and W be finite-dimensional vector spaces and $\beta: V \times V \to W$ a bilinear map. For the group*

$$\mathrm{O}(V,\beta) := \{g \in \mathrm{GL}(V) \colon (\forall v, v' \in V)\ \beta(g.v, g.v') = \beta(v,v')\}$$

we then have

$$\mathfrak{o}(V,\beta) := \mathbf{L}(\mathrm{O}(V,\beta)) = \{x \in \mathfrak{gl}(V) \colon (\forall v, v' \in V)\ \beta(x.v, v') + \beta(v, x.v') = 0\}.$$

Proof. We only have to observe that $X \in \mathbf{L}(\mathrm{O}(V,\beta))$ is equivalent to the pair $(X, 0)$ satisfying condition (1) in Lemma III.2.2. ■

Example III.2.4. (a) Let $B \in M_n(\mathbb{K})$, $\beta(v,w) = v^\top B w$, and

$$G := \{g \in \mathrm{GL}_n(\mathbb{K}) \colon g^\top B g = B\} \cong \mathrm{O}(\mathbb{K}^n, \beta).$$

Then Corollary III.2.3 implies that

$$\begin{aligned}\mathbf{L}(G) &= \{x \in \mathfrak{gl}_n(\mathbb{K}) \colon (\forall v, v' \in V)\ \beta(x.v, v') + \beta(v, x.v') = 0\}\\ &= \{x \in \mathfrak{gl}_n(\mathbb{K}) \colon (\forall v, v' \in V)\ v^\top x^\top B v' + v^\top B x v' = 0\}\\ &= \{x \in \mathfrak{gl}_n(\mathbb{K}) \colon x^\top B + Bx = 0\}.\end{aligned}$$

In particular, we obtain

$$\mathfrak{o}_n(\mathbb{K}) := \mathbf{L}(\mathrm{O}_n(\mathbb{K})) = \{x \in \mathfrak{gl}_n(\mathbb{K}) \colon x^\top = -x\} =: \mathrm{Skew}_n(\mathbb{K}),$$
$$\mathfrak{o}_{p,q}(\mathbb{K}) := \mathbf{L}(\mathrm{O}_{p,q}(\mathbb{K})) = \{x \in \mathfrak{gl}_{p+q}(\mathbb{K}) \colon x^\top I_{p,q} + I_{p,q} x = 0\},$$

and

$$\mathfrak{sp}_{2n}(\mathbb{K}) := \mathbf{L}(\mathrm{Sp}_{2n}(\mathbb{K})) := \{x \in \mathfrak{gl}_{2n}(\mathbb{K}) \colon x^\top B + Bx = 0\},$$

where $B = \begin{pmatrix} 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix}$.

(b) Applying Proposition III.2.3 with $V = \mathbb{C}^n$ and $W = \mathbb{C}$, considered as real vector spaces, we also obtain for a hermitian form $\beta: \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}, (z,w) \mapsto w^* I_{p,q} z$:

$$\begin{aligned}\mathfrak{u}_{p,q}(\mathbb{C}) &:= \mathbf{L}(\mathrm{U}_{p,q}(\mathbb{C}))\\ &= \{x \in \mathfrak{gl}_n(\mathbb{C}) \colon (\forall z, w \in \mathbb{C}^n)\, w^* I_{p,q} x z + w^* x^* I_{p,q} z = 0\}\\ &= \{x \in \mathfrak{gl}_n(\mathbb{C}) \colon I_{p,q} x + x^* I_{p,q} = 0\}.\end{aligned}$$

In particular we get

$$\mathfrak{u}_n(\mathbb{C}) := \mathbf{L}(\mathrm{U}_n(\mathbb{C})) = \{x \in \mathfrak{gl}_n(\mathbb{C}) \colon x^* = -x\}.$$ ■

Example III.2.5. Let $\mathfrak{g}$ be a finite-dimensional $\mathbb{K}$-Lie algebra and

$$\mathrm{Aut}(\mathfrak{g}) := \{g \in \mathrm{GL}(\mathfrak{g}) \colon (\forall x, y \in \mathfrak{g})\ g.[x,y] = [g.x, g.y]\}.$$

To calculate the Lie algebra of G, we use Lemma III.2.2 with $V = W = \mathfrak{g}$ and $\beta(x,y) = [x,y]$. Then we see that $D \in \mathfrak{aut}(\mathfrak{g}) := \mathbf{L}(\mathrm{Aut}(\mathfrak{g}))$ is equivalent to (D, D) satisfying the conditions in Lemma III.2.2, and this leads to

$$\mathfrak{aut}(\mathfrak{g}) = \mathbf{L}(\mathrm{Aut}(\mathfrak{g})) = \{D \in \mathfrak{gl}(\mathfrak{g}) \colon (\forall x, y \in \mathfrak{g})\, D.[x,y] = [D.x, y] + [x, D.y]\}.$$

The elements of this Lie algebra are called *derivations of* $\mathfrak{g}$, and $\mathfrak{aut}(\mathfrak{g})$ is also denoted $\mathrm{der}(\mathfrak{g})$. Note that the condition on an endomorphism of $\mathfrak{g}$ to be a derivation ressembles the Leibniz Rule (Product Rule). ■

Remark III.2.6. We call a linear Lie group $G \subseteq \mathrm{GL}_n(\mathbb{C})$ a *complex linear Lie group* if $\mathbf{L}(G) \subseteq \mathfrak{gl}_n(\mathbb{C})$ is a complex subspace, i.e., $i\,\mathbf{L}(G) \subseteq \mathbf{L}(G)$. Since Proposition II.3.4 only ensures that $\mathbf{L}(G)$ is a real subspace, this definition makes sense.

For example $\mathrm{U}_n(\mathbb{C})$ is not a complex linear Lie group because

$$i\mathfrak{u}_n(\mathbb{C}) = \mathrm{Herm}_n(\mathbb{C}) \not\subseteq \mathfrak{u}_n(\mathbb{C}).$$

On the other hand $\mathrm{O}_n(\mathbb{C})$ is a complex linear Lie group because

$$\mathfrak{o}_n(\mathbb{C}) = \mathrm{Skew}_n(\mathbb{C})$$

is a complex subspace of $\mathfrak{gl}_n(\mathbb{C})$. ∎

Polar decomposition of certain algebraic Lie groups

In this subsection we shall show that the polar decomposition of $\mathrm{GL}_n(\mathbb{R})$ can be used to obtain polar decompositions of many subgroups.

Let $G \subseteq \mathrm{GL}_n(\mathbb{K})$ be a linear Lie group. If $g = ue^x \in G$ (u unitary and x hermitian) implies that $u \in G$ and $e^x \in G$, then $g^* = e^x u^{-1} \in G$. Therefore a necessary condition for G to be adapated to the polar decomposition of G is that G is invariant under the map $g \mapsto g^*$. So we assume that this condition is satisfied. For $x \in \mathbf{L}(G)$ we then obtain from $(e^{tx})^* = e^{tx^*}$ that $x^* \in \mathbf{L}(G)$. Hence each element $x \in \mathbf{L}(G)$ can be written as $x = \frac{1}{2}(x - x^*) + \frac{1}{2}(x + x^*)$, where both summands are in $\mathbf{L}(G)$. This implies that

$$\mathbf{L}(G) = \mathfrak{k} \oplus \mathfrak{p}, \quad \text{where} \quad \mathfrak{k} := \mathbf{L}(G) \cap \mathfrak{u}_n(\mathbb{K}), \quad \mathfrak{p} := \mathbf{L}(G) \cap \mathrm{Herm}_n(\mathbb{K}).$$

We also need a condition which ensures that $e^x \in G$, $x \in \mathrm{Herm}_n(\mathbb{K})$, implies $x \in \mathbf{L}(G)$.

Definition III.2.7. We call a subgroup $G \subseteq \mathrm{GL}_n(\mathbb{R})$ *algebraic* if there exists a family $(p_j)_{j \in J}$ of real polynomials

$$p_j(x) = p_j(x_{11}, x_{12}, \ldots, x_{nn}) \in \mathbb{R}[x_{11}, \ldots, x_{nn}]$$

in the entries of the matrix $x \in M_n(\mathbb{R})$ such that

$$G = \{x \in \mathrm{GL}_n(\mathbb{R}) \colon (\forall j \in J)\ p_j(x) = 0\}.$$ ∎

Lemma III.2.8. *Let $G \subseteq \mathrm{GL}_n(\mathbb{R})$ be an algebraic subgroup, $y \in M_n(\mathbb{R})$ diagonalizable and $e^y \in G$. Then $y \in \mathbf{L}(G)$, i.e., $e^{\mathbb{R}y} \subseteq G$.*

Proof. Suppose that $A \in \mathrm{GL}_n(\mathbb{R})$ is such that AyA^{-1} is a diagonal matrix. Then $\widetilde{p}_j(x) = p_j(A^{-1}xA)$, $j \in J$, is also a set of polynomials in the entries of x and $e^y \in G$ is equivalent to

$$e^{AyA^{-1}} = Ae^yA^{-1} \in \widetilde{G} := AGA^{-1} = \{g \in \mathrm{GL}_n(\mathbb{R}) \colon (\forall j)\, \widetilde{p}_j(g) = 0\}.$$

Therefore we may assume that $y = \mathrm{diag}(y_1, \ldots, y_n)$ is a diagonal matrix. Now the polynomial $q_j(t) := p_j(e^{ty})$ has the form

$$q_j(t) = \sum_{(k_1,\ldots,k_n)\in\mathbb{N}_0^n} a_{k_1,\ldots,k_n} (e^{ty_1})^{k_1} \cdots (e^{ty_n})^{k_n}$$
$$= \sum_{(k_1,\ldots,k_n)\in\mathbb{N}_0^n} a_{k_1,\ldots,k_n} e^{t(k_1y_1+\ldots+k_ny_n)}$$

(a finite sum). Therefore it can be written as

$$q_j(t) = \sum_{k=1}^m \lambda_k e^{tb_k}, \quad b_1 > \ldots > b_m,$$

where each b_k is a sum of the entries y_k of y.

If q_j does not vanish on $\mathbb{R}$, then we may assume that $\lambda_1 \neq 0$. This leads to

$$\lim_{t\to\infty} e^{-tb_1} q_j(t) = \lambda_1 \neq 0,$$

which contradicts $q_j(\mathbb{Z}) = \{0\}$, which in turn follows from $e^{\mathbb{Z}y} \subseteq G$. Therefore each polynomial q_j vanishes, and hence $e^{\mathbb{R}y} \subseteq G$. ∎

Proposition III.2.9. *Let $G \subseteq \mathrm{GL}_n(\mathbb{R})$ be an algebraic subgroup with $G = G^\top$. We define $K := G \cap \mathrm{O}_n(\mathbb{R})$ and $\mathfrak{p} := \mathbf{L}(G) \cap \mathrm{Sym}_n(\mathbb{R})$. Then the map*

$$\varphi\colon K \times \mathfrak{p} \to G, \quad (k, x) \mapsto ke^x$$

is a homeomorphism.

Proof. Let $g \in G$ and write it as $g = ue^x$ with $u \in \mathrm{O}_n(\mathbb{R})$ and $x \in \mathrm{Sym}_n(\mathbb{R})$ (Proposition II.3.5 and the polar decomposition). Then

$$e^{2x} = g^\top g \in G,$$

where $x \in \mathrm{Sym}_n(\mathbb{R})$ is diagonalizable. Therefore Lemma III.2.8 implies that $e^{\mathbb{R}x} \subseteq G$, so that $x \in \mathfrak{p}$. Hence $u = ge^{-x} \in G \cap \mathrm{O}_n(\mathbb{R}) = K$. We conclude that φ is a surjective map. Furthermore Proposition I.1.4 on the polar decomposition of $\mathrm{GL}_n(\mathbb{R})$ implies that φ is injective, hence bijective. The continuity of φ^{-1} also follows from Proposition I.1.4. ∎

Examples III.2.10. Proposition III.2.9 applies to the following groups:

(a) $G = \mathrm{SL}_n(\mathbb{R})$ is $p^{-1}(0)$ for the polynomial $p(x) = \det x - 1$, and we obtain

$$\mathrm{SL}_n(\mathbb{R}) = K \exp \mathfrak{p} \cong K \times \mathfrak{p}$$

with

$$K = \mathrm{SO}_n(\mathbb{R}) \quad \text{and} \quad \mathfrak{p} = \{x \in \mathrm{Sym}_n(\mathbb{R})\colon \mathrm{tr}\, x = 0\}.$$

For the group $\mathrm{SL}_2(\mathbb{R})$ we obtain in particular a homeomorphism

$$\mathrm{SL}_2(\mathbb{R}) \cong \mathrm{SO}_2(\mathbb{R}) \times \mathbb{R}^2 \cong \mathbb{S}^1 \times \mathbb{R}^2.$$

(b) $G = \mathrm{O}_{p,q} := \mathrm{O}_{p,q}(\mathbb{R})$ is defined by the condition $g^\top I_{p,q} g = I_{p,q}$. These are n^2 polynomial equations, one for each entry of the matrix. Moreover, $g \in \mathrm{O}_{p,q}$ implies

$$I_{p,q} = I_{p,q}^{-1} = (g^\top I_{p,q} g)^{-1} = g^{-1} I_{p,q} (g^\top)^{-1}$$

and hence $g I_{p,q} g^\top = I_{p,q}$, i.e., $g^\top \in \mathrm{O}_{p,q}$. Therefore $\mathrm{O}_{p,q}^\top = \mathrm{O}_{p,q}$, and all the assumptions of Proposition III.2.9 are satisfied. In this case we have

$$K = \mathrm{O}_{p,q} \cap \mathrm{O}_n \cong \mathrm{O}_p \times \mathrm{O}_q,$$

(Exercise III.2.6) and topologically we obtain

$$\mathrm{O}_{p,q} \cong \mathrm{O}_p \times \mathrm{O}_q \times (\mathfrak{o}_{p,q} \cap \mathrm{Sym}_n(\mathbb{R})).$$

In particular we see that for $p, q > 0$ the group $\mathrm{O}_{p,q}$ has four arc-components because O_p and O_q have two arc-components (Proposition I.1.6).

For the subgroup $\mathrm{SO}_{p,q}$ we have one additional polynomial equation, so that it is also algebraic. Here we have

$$\begin{aligned} K_S := K \cap \mathrm{SO}_{p,q} &\cong \{(a,b) \in \mathrm{O}_p \times \mathrm{O}_q \colon \det(a)\det(b) = 1\} \\ &\cong \big(\mathrm{SO}_p \times \mathrm{SO}_q\big) \dot\cup \big(\mathrm{O}_{p,-} \times \mathrm{O}_{q,-}\big), \end{aligned}$$

so that $\mathrm{SO}_{p,q}$ has two arc-components if $p, q > 0$ (cf. the discussion of the Lorentz group in Example I.2.7).

(c) We can also apply Proposition III.2.9 to the subgroup $\mathrm{GL}_n(\mathbb{C}) \subseteq \mathrm{GL}_{2n}(\mathbb{R})$ which is defined by the condition $gI = Ig$, where $I \colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ corresponds to the componentwise multiplication with i on $\mathbb{C}^n$. These are $4n^2 = (2n)^2$ polynomial equations defining $\mathrm{GL}_n(\mathbb{C})$. In this case we obtain a new proof of the polar decomposition of $\mathrm{GL}_n(\mathbb{C})$ because

$$K = \mathrm{GL}_n(\mathbb{C}) \cap \mathrm{O}_{2n}(\mathbb{R}) = \mathrm{U}_n(\mathbb{C})$$

and

$$\mathfrak{p} = \mathfrak{gl}_n(\mathbb{C}) \cap \mathrm{Sym}_{2n}(\mathbb{R}) = \mathrm{Herm}_n(\mathbb{C}). \qquad \blacksquare$$

Example III.2.11. Let $X \in \mathrm{Sym}_n(\mathbb{R})$ be a non-zero symmetric matrix and consider the subgroup $G := \exp(\mathbb{Z}X) \subseteq \mathrm{GL}_n(\mathbb{R})$. Since $\exp X$ is symmetric, we then have $G^\top = G$. Moreover, if $\lambda_1 \leq \ldots \leq \lambda_k$ are the eigenvalues of X, then

$$\|\exp(nX) - \mathbf{1}\| = \max(|e^{n\lambda_k} - 1|, |e^{n\lambda_1} - 1|) \geq \max(|e^{\lambda_k} - 1|, |e^{\lambda_1} - 1|)$$

implies that G is a discrete subset of $\mathrm{GL}_n(\mathbb{R})$, hence a closed subgroup, and therefore a linear Lie group. On the other hand, the fact that G is discrete implies that $\mathbf{L}(G) = \{0\}$. This example show that the assumption that G is algebraic is indispensable for Proposition III.2.9 because

$$G \cap O_n(\mathbb{R}) = \{\mathbf{1}\} \quad \text{and} \quad \mathbf{L}(G) \cap \mathrm{Sym}_n(\mathbb{R}) = \{0\}. \qquad \blacksquare$$

Exercises for Section III.2.

Exercise III.2.1. Show that the groups $\mathrm{O}_n(\mathbb{C})$, $\mathrm{SO}_n(\mathbb{C})$ and $\mathrm{Sp}_{2n}(\mathbb{R})$ have polar decompositions and describe their intersections with $\mathrm{O}_{2n}(\mathbb{R})$. ■

Exercise III.2.2. Let $B \in \mathrm{Herm}_n(\mathbb{K})$ with $B^2 = \mathbf{1}$ and consider the automorphism $\tau(g) = Bg^{-\top}B^{-1}$ of $\mathrm{GL}_n(\mathbb{K})$.

(1) $\mathrm{O}(\mathbb{C}^n, B) = \{g \in \mathrm{GL}_n(\mathbb{K}) \colon \tau(g) = g\}$.

(2) Show that $\mathrm{O}(\mathbb{C}^n, B)$ is adapted to the polar decomposition by showing that if $g = ue^x$ is the polar decomposition of g, then $\tau(g) = g$ is equivalent to $\tau(u) = u$ and $\tau(x) = x$.

(3) Show that $\mathrm{O}(\mathbb{C}^n, B)$ is adapted to the polar decomposition by using that it is an algebraic group. ■

Exercise III.2.3. Show that the following groups are linear Lie groups and determine their Lie algebras.

(1) $N := \{g \in \mathrm{GL}_n(\mathbb{R}) \colon (\forall i > j)\ g_{ij} = 0, g_{ii} = 1\}$.
(2) $B := \{g \in \mathrm{GL}_n(\mathbb{R}) \colon (\forall i > j)\ g_{ij} = 0\}$.
(3) $D := \{g \in \mathrm{GL}_n(\mathbb{R}) \colon (\forall i \neq j)\ g_{ij} = 0\}$.
(4) A a finite-dimensional associative algebra and

$$G := \mathrm{Aut}(A) := \{g \in \mathrm{GL}(A) \colon (\forall a, b \in A)\ g(ab) = g(a)g(b)\}.$$

Note that $B \cong N \rtimes D$ is a semidirect product. ■

Exercise III.2.4. Realize the two groups $\mathrm{Mot}_n(\mathbb{R})$ and $\mathrm{Aff}_n(\mathbb{R})$ as linear Lie groups in $\mathrm{GL}_{n+1}(\mathbb{R})$.

(1) Determine their Lie algebras $\mathfrak{mot}_n(\mathbb{R})$ and $\mathfrak{aff}(n, \mathbb{R})$.
(2) Calculate the exponential function $\exp \colon \mathfrak{aff}_n(\mathbb{R}) \to \mathrm{Aff}_n(\mathbb{R})$ in terms of the exponential function of $M_n(\mathbb{R})$. ■

Exercise III.2.5. Let V be a finite-dimensional $\mathbb{K}$-vector space and $W \subseteq V$ a subspace. Show that

$$\mathrm{GL}(V, W) := \{g \in \mathrm{GL}(V) \colon g.W = W\}$$

is a closed subgroup of $\mathrm{GL}(V)$ with

$$\mathbf{L}(\mathrm{GL}(V, W)) = \mathfrak{gl}(V, W) := \{X \in \mathfrak{gl}(V) \colon X.W \subseteq W\}.$$ ■

Exercise III.2.6. Show that for $n = p + q$ we have

$$\mathrm{O}_{p,q}(\mathbb{K}) \cap \mathrm{O}_n(\mathbb{K}) \cong \mathrm{O}_p(\mathbb{K}) \times \mathrm{O}_q(\mathbb{K}).$$ ■

III.3. Local properties of linear Lie groups

In the preceding section we have seen techniques permitting us to calculate the Lie algebra for a concrete group. In this section we turn to the converse question. To which extent does the Lie algebra $\mathbf{L}(G)$ determine the structure of the group G? The main result of this chapter will be that it completely determines the local structure of G, i.e., the group multiplication in a sufficiently small neighborhood of the identity. Our strategy is to show first that the exponential function $\exp_G: \mathbf{L}(G) \to G$ restricts to a homeomorphism of a 0-neighborhood in $\mathbf{L}(G)$ onto a $\mathbf{1}$-neighborhood in G. In view of the results of Section II.4 on the Dynkin series, this implies that the group multiplication (in a small $\mathbf{1}$-neighborhood) is explicitly determined by the Lie algebra structure of $\mathbf{L}(G)$ because it is given by the locally convergent Dynkin series.

The local structure of linear Lie groups

The goal of this subsection is the following theorem.

THE IDENTITY NEIGHBORHOOD THEOREM

Theorem III.3.1. *Let $G \subseteq \mathrm{GL}_n(\mathbb{K})$ be a closed subgroup. Then each 0-neighborhood in $\mathbf{L}(G)$ contains an open 0-neighborhood $V \subseteq \mathbf{L}(G)$ such that $\exp|_V: V \to W := \exp(V) \subseteq G$ is a homeomorphism onto an open subset of G.*

Proof. First we use Proposition II.2.4 to find an open 0-neighborhood $V_o \subseteq \mathfrak{gl}_n(\mathbb{K})$ such that

$$\exp_{V_o} := \exp|_{V_o}: V_o \to W_o := \exp(V_o)$$

is a diffeomorphism between open sets. In the following we write $\log_{W_o} := (\exp_{V_o})^{-1}$ for the inverse function. Then the following assertions hold:

- $V_o \cap \mathbf{L}(G)$ is a 0-neighborhood in $\mathbf{L}(G)$.
- $W_o \cap G$ is a $\mathbf{1}$-neighborhood in G.
- $\exp(V_o \cap \mathbf{L}(G)) \subseteq W_o \cap G$
- $\exp|_{V_o \cap \mathbf{L}(G)}$ is injective.

If G is not closed, then it need not be true that

$$\exp(V_o \cap \mathbf{L}(G)) = W_o \cap G$$

because it might be the case that $W_o \cap G$ is much larger than $\exp(V_o \cap \mathbf{L}(G))$ (see "the dense wind" discussed in a separate subsection below). We do not even know whether $\exp(V_o \cap \mathbf{L}(G))$ is open in G. Before we can complete the proof, we need three lemmas.

Lemma III.3.2. *Let $(g_k)_{k\in\mathbb{N}}$ be a sequence in $G \cap W_o$ with $g_k \neq \mathbf{1}$ for all $k \in \mathbb{N}$ and $g_k \to \mathbf{1}$. We put $y_k := \log_{W_o} g_k$. Then every cluster point of the sequence*

$$\Big\{ \frac{y_k}{\|y_k\|} : k \in \mathbb{N} \Big\}$$

is contained in $\mathbf{L}(G)$.

Proof. Let x be such a cluster point. By replacing the original sequence by a subsequence, we may assume that

$$x_k := \frac{y_k}{\|y_k\|} \to x \in \mathfrak{gl}_n(\mathbb{K})$$

and note that this implies $\|x\| = 1$. Let $t \in \mathbb{R}$ and put $p_k := \frac{t}{\|y_k\|}$. Then $tx_k = p_k y_k$, $y_k \to \log_{W_o} \mathbf{1} = 0$,

$$\exp tx = \lim_{k\to\infty} \exp(tx_k) = \lim_{k\to\infty} \exp(p_k y_k),$$

and

$$\exp(p_k y_k) = \exp(y_k)^{[p_k]} \exp\big((p_k - [p_k])y_k\big),$$

where $[p_k] = \max\{l \in \mathbb{Z} : l \leq p_k\}$ is the *Gauß function.* We therefore have

$$\|(p_k - [p_k])y_k\| \leq \|y_k\| \to 0$$

and eventually

$$\exp tx = \lim_{k\to\infty} (\exp y_k)^{[p_k]} = \lim_{k\to\infty} g_k^{[p_k]} \in G,$$

because G is closed. This implies $x \in \mathbf{L}(G)$. ■

Lemma III.3.3. *Let $E \subseteq \mathfrak{gl}_n(\mathbb{K})$ be a real vector subspace complementing the real subspace $\mathbf{L}(G)$. Then there exists a 0-neighborhood $U_E \subseteq E$ with*

$$G \cap \exp U_E = \{\mathbf{1}\}.$$

Proof. We argue by contradiction. If a neighborhood U_E with the required properties does not exist, then for each compact convex 0-neighborhood $V_E \subseteq E$ we have for each $k \in \mathbb{N}$:

$$(\exp \tfrac{1}{k} V_E) \cap G \neq \{\mathbf{1}\}.$$

For each $k \in \mathbb{N}$ we therefore find a $y_k \in V_E$ with $\mathbf{1} \neq g_k := \exp \frac{y_k}{k} \in G$. Now the compactness of V_E implies that the sequence $(y_k)_{k\in\mathbb{N}}$ is bounded, so that $\frac{y_k}{k} \to 0$, which implies $g_k \to \mathbf{1}$. Now let $x \in E$ be a cluster point of the sequence $\frac{y_k}{\|y_k\|}$ which lies in the compact set $S_E := \{z \in E : \|z\| = 1\}$, so that at least one cluster point exists. According to Lemma III.3.2, we have $x \in \mathbf{L}(G) \cap E = \{0\}$ because Lemma III.3.2 applies since $g_k \in G \cap W_o$ for k sufficiently large. We arrive at a contradiction to $\|x\| = 1$. This proves the lemma. ■

Lemma III.3.4. *Let* $V_1, \ldots, V_r \subseteq \mathfrak{gl}_n(\mathbb{K})$ *be vector subspaces with*

$$V_1 \oplus \ldots \oplus V_r = \mathfrak{gl}_n(\mathbb{K}).$$

Then the map

$$\Phi\colon V_1 \times \ldots \times V_r \to \mathrm{GL}_n(\mathbb{K}), \quad (x_1, \ldots, x_r) \mapsto (\exp x_1) \ldots (\exp x_r)$$

restricts to a diffeomorphism of a neighborhood of $(0, \ldots, 0)$ *to an open* $\mathbf{1}$*-neighborhood in* $\mathrm{GL}_n(\mathbb{K})$*.*

Proof. Let $\mu_r\colon M_n(\mathbb{K}) \times \ldots \times M_n(\mathbb{K}) \to M_n(\mathbb{K})$ be the multiplication map

$$(x_1, \ldots, x_r) \mapsto x_1 \ldots x_r.$$

This map is r-linear, so that its derivative is given by

$$\mathtt{d}\mu_r(x_1, \ldots, x_r)(y_1, \ldots, y_r) = y_1 x_2 \cdots x_r + \ldots + x_1 \cdots x_{r-1} y_r$$

(Lemma A.2.6). In particular we have

$$\mathtt{d}\mu_r(\mathbf{1}, \ldots, \mathbf{1})(y_1, \ldots, y_r) = y_1 + \ldots + y_r.$$

Now the Chain Rule and $\mathtt{d}\exp(0) = \mathrm{id}$ lead to

$$\begin{aligned}\mathtt{d}\Phi(0, \ldots, 0)(y_1, \ldots, y_r) &= \mathtt{d}\mu_r(\mathbf{1}, \ldots, \mathbf{1})(\mathtt{d}\exp(0).y_1, \ldots, \mathtt{d}\exp(0).y_r) \\ &= \mathtt{d}\mu_r(\mathbf{1}, \ldots, \mathbf{1})(y_1, \ldots, y_r) = y_1 + \ldots + y_r.\end{aligned}$$

Since this map is bijective, the Inverse Function Theorem implies that Φ restricts to a diffeomorphism of a 0-neighborhood in $V_1 \times \ldots \times V_r$ onto a $\mathbf{1}$-neighborhood in $\mathrm{GL}_n(\mathbb{K})$. ■

Now we are ready to complete the proof of Theorem III.3.1. We choose E as above, a vector space complement to $\mathbf{L}(G)$, and define

$$\Phi\colon \mathbf{L}(G) \times E \to \mathrm{GL}_n(\mathbb{K}), \quad (x, y) \mapsto \exp x \exp y.$$

According to Lemma III.3.4, there exist open 0-neighborhoods $U_E \subseteq E$ and $U_o \subseteq \mathbf{L}(G)$ such that

$$\Phi|_{U_o \times U_E}\colon U_o \times U_E \to \exp(U_o)\exp(U_E)$$

is a diffeomorphism onto an open $\mathbf{1}$-neighborhood in $\mathrm{GL}_n(\mathbb{K})$. Moreover, in view of Lemma III.3.3 we may choose U_E so small that $\exp(U_E) \cap G = \{\mathbf{1}\}$.

Since $\exp(U_o) \subseteq G$, the condition $g = \exp x \exp y \in G \cap (\exp(U_o)\exp(U_E))$ implies $\exp y = (\exp x)^{-1} g \in G \cap \exp U_E = \{\mathbf{1}\}$. Therefore

$$\exp(U_o) = G \cap (\exp(U_o)\exp(U_E))$$

is an open $\mathbf{1}$-neighborhood in G. This completes the proof of Theorem III.3.1. ■

Linear Lie groups as submanifolds

The Identity Neighborhood Theorem has important consequences for the structure of linear Lie groups. One of them is that they are submanifolds of the real vector space $M_n(\mathbb{K}) \cong \mathbb{K}^{(n^2)}$.

Definition III.3.5. Let V be a finite-dimensional real vector space. A subset $M \subseteq V$ is called a *k-dimensional submanifold* if for each $x \in M$ there exists an open neighborhood U_x of x in V, a k-dimensional subspace $F \subseteq V$ and a diffeomorphism $\varphi: U_x \to W$ onto an open neighborhood W of 0 in V such that

$$\varphi(U_x \cap M) = W \cap F.$$

Geometrically this means that a piece of M (such as $U_x \cap M$) looks like a piece of a vector subspace F of V (such as $W \cap F$). In this sense φ "straightens" the curved structure of M. ■

Proposition III.3.6. *Every closed subgroup G of $\mathrm{GL}_n(\mathbb{K})$ is a submanifold of $M_n(\mathbb{K})$ of dimension $\dim_{\mathbb{R}} \mathbf{L}(G)$.*

Proof. We recall the diffeomorphism

$$\Phi: U_o \times U_E \to \exp(U_o)\exp(U_E)$$

from the proof of Theorem III.3.1, where $U_o \subseteq \mathbf{L}(G)$ and $U_E \subseteq E$ are open 0-neighborhoods and $M_n(\mathbb{K}) = \mathbf{L}(G) \oplus E$. We also recall that

$$\Phi(U_o \times U_E) \cap G = \exp(U_o) = \Phi(U_o \times \{0\}).$$

For $g \in G$ we write $\lambda_g: \mathrm{GL}_n(\mathbb{K}) \to \mathrm{GL}_n(\mathbb{K})$ for the left multiplication $\lambda_g(h) = gh$ and observe that λ_g is a linear automorphism of $M_n(\mathbb{K})$. Therefore $U_g := \lambda_g(\mathrm{im}(\Phi)) = g\,\mathrm{im}(\Phi)$ is an open neighborhood of g in $M_n(\mathbb{K})$. Moreover, the map

$$\varphi_g: U_g \to \mathbf{L}(G) \oplus E = M_n(\mathbb{K}), \quad x \mapsto \Phi^{-1}(g^{-1}x)$$

is a diffeomorphism onto the open subset $U_o \times U_E$ of $M_n(\mathbb{K})$, and we have

$$\begin{aligned}\varphi_g(U_g \cap G) &= \varphi_g(g\,\mathrm{im}(\Phi) \cap G) = \varphi_g(g(\mathrm{im}(\Phi) \cap G)) \\ &= \varphi_g(g\exp(U_o)) = U_o \times \{0\} = (U_o \times U_E) \cap (\mathbf{L}(G) \times \{0\}).\end{aligned}$$

Therefore the family $(\varphi_g, U_g)_{g \in G}$ satisfies the assumptions of Definition III.3.5, so that G is a submanifold of $M_n(\mathbb{K})$ of dimension $\dim_{\mathbb{R}} \mathbf{L}(G)$. ■

Remark III.3.7. (a) Every submanifold M of a vector space V is *locally closed* in the sense that for each $x \in M$ there exists a neighborhood U of x in V for which $U \cap M$ is closed in U.

(b) One can show that each locally closed subgroup H of a topological group G is closed. Therefore each subgroup $G \subseteq \mathrm{GL}_n(\mathbb{K})$ which is a submanifold of $M_n(\mathbb{K})$ is automatically closed, hence a linear Lie group. This means that the linear Lie groups are precisely those subgroups of $\mathrm{GL}_n(\mathbb{K})$ which are submanifolds of $M_n(\mathbb{K})$.

(c) For each submanifold $M \subseteq V$ and each $x \in M$ we define the *geometric tangent space* $T_x(M) \subseteq V$ as the set of all $v \in V$ for which there exists a differentiable curve $\gamma\colon]-\varepsilon, \varepsilon[\to M \subseteq V$ with $\gamma(0) = x$ and $\gamma'(0) = v$. In terms of Definition III.3.5, it is not hard to see that $T_x(M) = d\varphi(x)^{-1}(E)$. In particular $T_x M$ is a k-dimensional vector subspace of V.

(d) If G is a linear Lie group, then

$$T_{\mathbf{1}}(G) = \mathbf{L}(G).$$

In fact, $\gamma(t) := \exp tx \subseteq G$ for $x \in \mathfrak{g}$, $t \in \mathbb{R}$, and Lemma II.2.2 imply that $x = \gamma'(0) \in T_{\mathbf{1}}(G)$ and hence $\mathbf{L}(G) \subseteq T_{\mathbf{1}}(G)$. Since the spaces $\mathbf{L}(G)$ and $T_{\mathbf{1}}(G)$ have the same dimension (Proposition III.3.6), both are equal. ■

The dense wind

In this short subsection we discuss an important example of a subgroup of $\mathrm{GL}_2(\mathbb{C})$ which is not closed and therefore not a submanifold. It is the simplest example of a non-closed, arcwise connected subgroup of $\mathrm{GL}_n(\mathbb{K})$.

Let

$$A = \left\{ \begin{pmatrix} e^{it\sqrt{2}} & 0 \\ 0 & e^{it} \end{pmatrix} : t \in \mathbb{R} \right\} \subseteq \mathbb{T}^2 := \left\{ \begin{pmatrix} e^{ir} & 0 \\ 0 & e^{is} \end{pmatrix} : r, s \in \mathbb{R} \right\},$$

where $\mathbb{T}^2$ is the *two-dimensional torus.* It is clear that $\mathbb{T}^2$ is closed in $\mathrm{GL}_2(\mathbb{C})$, hence a linear Lie group.

Lemma III.3.8. *A is dense in $\mathbb{T}^2$.*

Proof. We consider the map

$$\Phi\colon \mathbb{R}^2 \to \mathbb{T}^2, \quad (r, s) \mapsto \begin{pmatrix} e^{2\pi ir} & 0 \\ 0 & e^{2\pi is} \end{pmatrix}$$

which is a surjective continuous group homomorphism with kernel $\mathbb{Z}^2$. For $L := \mathbb{R}(\sqrt{2}, 1)$ and $V = \mathbb{R}(1, 0)$ we have $\mathbb{R}^2 \cong V \oplus L$. In view of

$$A = \Phi(L) = \Phi(L + \mathbb{Z}^2),$$

it suffices to show that $L + \mathbb{Z}^2$ is dense in $\mathbb{R}^2$. From the direct decomposition $\mathbb{R}^2 \cong V \oplus L$ and $L \subseteq L + \mathbb{Z}^2$ we derive

$$L + \mathbb{Z}^2 = L + ((L + \mathbb{Z}^2) \cap V),$$

and if $p\colon \mathbb{R}^2 \to V$ denote the projection map with kernel L, then

$$(L + \mathbb{Z}^2) \cap V = p(L + \mathbb{Z}^2) = p(\mathbb{Z}^2).$$

It therefore suffices to show that $p(\mathbb{Z}^2)$ is dense in V. From $p(1,0) = (1,0)$ and $p(0,1) = p((0,1) - (\sqrt{2}, 1)) = -(\sqrt{2}, 0)$ we obtain $p(\mathbb{Z}^2) = \mathbb{Z} + \sqrt{2}\mathbb{Z}$, so that the density of $p(\mathbb{Z}^2)$ is a consequence of the following lemma. ■

Lemma III.3.9. *Let $r \in \mathbb{R}$. Then $\mathbb{Z} + r\mathbb{Z}$ is dense in $\mathbb{R}$ if and only if r is irrational.*

Proof. We call a subgroup $H \subseteq \mathbb{R}$ discrete if there exists a 0-neighborhood $U \subseteq \mathbb{R}$ with $U \cap H = \{0\}$.

Step 1: Each discrete subgroup $H \subseteq \mathbb{R}$ is cyclic: Let $H \subseteq \mathbb{R}$ be a non-zero discrete subgroup. Then

$$x_o := \inf\{x \in H : x > 0\} > 0$$

and for $x < y \in H$ we have $y - x \geq x_o$. The definition of x_o implies the existence of $x \in H$ with $x_o \leq x < 2x_o$. If $x \neq x_o$, then there exists $y \in H$ with $x_o \leq y < x$. Then $0 < x - y < x_o$, contradicting the definition of x_o. Therefore $x_o = x \in H$ and further $\mathbb{Z}x_o \subseteq H$. For any element $z \in H$ we now find $k \in \mathbb{Z}$ with $z - kx_o \in H \cap [0, x_o[$, so that the definition of x_o leads to $z - kx_o = 0$. This proves that $H = \mathbb{Z}x_o$.

Step 2: If $H \subseteq \mathbb{R}$ is a subgroup which is not discrete, then it is dense: If H is not discrete, then it contains a sequence (x_k) with $0 \neq x_k \to 0$. For each $x \in \mathbb{R}$ and $k \in \mathbb{N}$ we further find $m_k \in \mathbb{Z}$ mit $|m_k x_k - x| \leq x_k$. Hence $(m_k x_k)_{k \in \mathbb{N}}$ is a sequence in H converging to x. This means that each non-discrete subgroup is dense.

Step 3: Suppose first that $\mathbb{Z} + r\mathbb{Z}$ is not dense in $\mathbb{R}$. Then it is discrete by Step 2, hence of the form $\mathbb{Z}x_o$ for some $x_o > 0$. Then there exist $k, m \in \mathbb{Z}$ with

$$1 = kx_o \quad \text{and} \quad r = mx_o$$

and we obtain $r = \frac{m}{k} \in \mathbb{Q}$. If, conversely, $r = \frac{m}{k} \in \mathbb{Q}$, then $\mathbb{Z} + r\mathbb{Z} \subseteq \frac{1}{k}\mathbb{Z}$ is not dense in $\mathbb{R}$. ■

Exercises for Section III.3.

The following three exercises discuss closed subgroups of $\mathbb{R}^n$.

Exercise III.3.1. Let $D \subseteq \mathbb{R}^n$ be a discrete subgroup. Then there exist linearly independent elements $v_1, \ldots, v_k \in \mathbb{R}^n$ with $D = \sum_{i=1}^k \mathbb{Z} v_i$. Hint: Use induction on $\dim(\operatorname{span} D)$. If $n > 1$, and D spans $\mathbb{R}^n$, then pick linearly independent elements $f_1, \ldots, f_{n-1} \in D$ and apply induction on $F \cap D$ for $F := \operatorname{span}\{f_1, \ldots, f_{n-1}\}$, where F is a hyperplane in $\mathbb{R}^n$. Now choose $f_n \in D$ with $D = \mathbb{Z} f_n + D \cap F$. This can be done by assuming that $F = \mathbb{R}^{n-1}$ and then choosing f_n with minimal positive n-th component (Verify the existence!). ■

Exercise III.3.2. The map

$$\Phi : \mathbb{R}^n \to \mathrm{GL}_{n+1}(\mathbb{R}), \quad (t_1, \ldots, t_n) \mapsto \begin{pmatrix} 1 & x_1 & \ldots & & x_n \\ 0 & 1 & 0 & \cdots & 0 \\ \cdot & & \cdot & & \cdot \\ 0 & & \ldots & & 1 \end{pmatrix}$$

is a homeomorphism and a group isomorphism. Hint: $\log : \Phi(\mathbb{R}^n) \to \mathbb{R}^n$ is continuous. ■

Exercise III.3.3. (Structure of closed subgroups of $\mathbb{R}^n$) Every closed subgroup $G \subseteq \mathbb{R}^n$ can be written as

$$G = F \oplus \mathbb{Z} f_1 \oplus \ldots \oplus \mathbb{Z} f_k,$$

where $F \subseteq \mathbb{R}^n$ is a vector subspace and $\dim \operatorname{span}(F \cup \{f_1, \ldots f_k\}) = \dim F + k$. Prove this via the following steps:

a) There exists a 0-neighborhood U in $\mathbb{R}^n$ with $U \cap G = F \cap U$, where $F \subseteq G$ is a maximal vector subspace. Hint: Theorem III.3.1 and Exercise III.3.2.

b) If $E \subseteq \mathbb{R}^n$ is a subspace with $\mathbb{R}^n = E \oplus F$, then $G = F \oplus (G \cap E)$.

c) $G \cap E = \mathbb{Z} f_1 \oplus \ldots \oplus \mathbb{Z} f_k$ for linearly independent elements $f_1, \ldots, f_k$ in E. Hint: $E \cap G$ is discrete. ■