Mathematics with Computer Science

Introductory Course
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Exercises Unit 3

1. Show by a direct proof that for all $a, b \in \mathbb{R}$ the equation $a + \frac{1}{a} = b$ implies $a^3 + \frac{1}{a^3} = b^3 - 3b$.

2. Show, that $\frac{3x-4}{2x+4} + 1 \leq 0$ implies $x \leq 0$.

3. (a) Let n be a natural number. Show that n^2 is even if and only if n is even.

   (b) Show that $x^2 = 6$ does not have a rational solution.

   (c) Show that $1 + \sqrt{2}$ is not a rational number. Show that $a + b\sqrt{2}$ is not rational for rational numbers a and b with $b \neq 0$.

   (d) Show that $x^3 = 2$ does not have a rational solution.

4. Prove by induction that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

5. Show, that for each $n \in \mathbb{N}$ with $n > 3$ the following is true:

   (a) $2n + 1 < n^2$,

   (b) $n^2 \leq 2^n$.

6. **What is wrong?**

Assume the following equation for a complex number x:

$$x^2 + x + 1 = 0.$$

Then

$$x^2 = -1 - x.$$

If we assume that $x \neq 0$, we can divide by x which yields to

$$x = -\frac{1}{x} - 1.$$

Substituting this expression in the original equation leads to

$$\begin{aligned} x^2 - \frac{1}{x} - 1 + 1 &= 0 \\ x^2 - \frac{1}{x} &= 0 \\ x^2 &= \frac{1}{x} \\ x^3 &= 1 \\ x &= 1. \end{aligned}$$

Substituting $x = 1$ in the original equation yields to

$$3 = 0.$$