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Fachbereich Mathematik
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Exercises Unit 1

NOTE: These exercises are to be solved in small teams of 3 to 4 students each. Discuss your solutions among each other. If you have questions, first see if one of your fellow students can answer your question, then ask your tutor. Teamwork is an important part of mathematics!

Rearrange the tables and chairs in the seminar room with your fellow students such that you can sit around the table, face each other and discuss. It is a good idea to put two or three tables together to form one large table.

Why don't you start now? In fact, this is the first thing you should do each time you come into the seminar room before you start working on the exercise problems!

1. Prove that 7 is not a divisor of 100.
2. List all the divisors of 12, 140 and 1001. Prove that your list of divisors for 12 is complete.
3. Use the Sieve of Eratosthenes to find all primes less than 50. What is the earliest point at which you have obtained all primes less than 50? Compare with exercise 6.
4. Prove that each natural number n is divisible by 1 and n.
5. Let p, q and r be natural numbers.

   Prove: If p is a divisor of q and q is a divisor of r, then p is a divisor of r.
6. Let n be a natural number.

   Prove: If $d|n$ with $d \geq \sqrt{n}$, then there is a divisor e of n with $e \leq \sqrt{n}$.

   How can you use this to simplify the test for primality by trial division?

   Compare this result with your observation about the Sieve of Eratosthenes.
7. Show: If d is a divisor of m and of n, then d is a divisor of $m+n$, $m-n$ and d^2 is a divisor of $m \cdot n$.
8. Calculate the follwing sums:

$$\sum_{i=0}^{3} i$$

$$\sum_{j=1}^{4} 3^j$$

$$\sum_{k=1}^{2} 2+j$$

9. Write with the sigma sign:

$$1 + 3 + 5 + 7 + 9$$
$$2 + 4 + 6 + 8 + 10$$
$$2 + 4 + 8 + 16$$

10. Show:

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

11. Show:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \qquad \text{and} \qquad \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

Hint: Use the Binomial Theorem.

12. Let a, b and c be real numbers and ε a positive real number.

    (a) Show that $|a| \leq c$ is the same as saying $-c \leq a \leq c$.

    (b) Show that $a \leq |a|$ and $-|a| \leq a$.

    (c) Prove the triangle inequality: $|a + b| \leq |a| + |b|$. Hint: Use the previous two inequalities.

    (d) Prove the inequality $|a| - |b| \leq |a - b|$.

    (e) Show that $|x - a| \leq \varepsilon$ is the same as saying $a - \varepsilon \leq x \leq a + \varepsilon$. Interpret this geometrically! What is the set of all x satisfying this condition?

    (f) Determine the solutions of the inequalities $|4 - 3x| > 2x + 10$ and $|2x - 10| \leq x$.