Technische Universität Darmstadt
Fachbereich Mathematik
Dennis Frisch

# Hints to Exercises Unit 8

1. (a) The normal form of a line is $y = ax + b$. The slope $a$ of the tangent line in $x_0$ is the derivative of $f$ in $x_0 = -1$. The parameter $b$ can be obtained from the equation $y = f(x_0) = 6x_0 + b$. We get $f'(x) = 6x^2$, $f'(-1) = 6$ and $f(-1) = -9$. The normal form of the tangent in $x_0 = -1$ is $y = 6x - 3$.

   (b) $f'(\frac{1}{2}) = -\frac{1}{\frac{1}{4}} = -4$, $f(\frac{1}{2}) = 2$. Then $2 = -4\frac{1}{2} + b$ gives $b = 4$ and the normal form of the tangent is $y = -4x + 4$.

2. Here we get

$$\frac{f(x) - f(0)}{x - 0} = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases}$$

   The difference quotient has the limit $0$ for $x \to 0$. Therefore, the function $f$ is differentiable.

3.

$$f'(x) = \frac{3(2x^2 + x)^2(4x + 1)}{2\sqrt{(2x^2 + x)^3 + 1}}$$

4.

$$\begin{aligned}
\lim_{x_n \to x_0} \frac{(f + g(x_n) - (f + g)(x_0)}{x_n - x_0} &= \lim_{x_n \to x_0} \frac{f(x_n) + g(x_n) - (f(x_0) + g(x_0))}{x_n - x_0} \\
&= \lim_{x_n \to x_0} \frac{f(x_n) - f(x_0) + g(x_n) - g(x_0)}{x_n - x_0} \\
&= \lim_{x_n \to x_0} \frac{f(x_n) - f(x_0)}{x_n - x_0} + \frac{g(x_n) - g(x_0)}{x_n - x_0} \\
&= \lim_{x_n \to x_0} \frac{f(x_n) - f(x_0)}{x_n - x_0} + \lim_{x_n \to x_0} \frac{g(x_n) - g(x_0)}{x_n - x_0} \\
&= f'(x_0) + g'(x_0)
\end{aligned}$$

5.

$$(\frac{f}{g}) = (fg^{-1}) = f'g^{-1} + f(g^{-1})' = f'g^{-1} + f(-g^{-2}g') = \frac{f'g - fg'}{g^2}$$

6. Maximize the function $f(x, y) = xy$ under the condition that $x + y = c$. This gives $f(x) = x(c - x) = xc - x^2$. Then $f'(x) = c - 2x$. An extremal point is at $x_0 = \frac{1}{2}c$. Since $f''(x_0) = -2$, this is a local maximum. Alternatively, show that $f(x) \leq f(\frac{1}{2}c)$ for all $x$:

$$f(x) \leq f(\frac{1}{2}c) \Leftrightarrow x(c - x) \leq \frac{1}{4}c^2 \Leftrightarrow 0 \leq x^2 - cx + \frac{1}{4}c^2 = (x - \frac{1}{2}c)^2.$$