Mathematics with Computer Science

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Introductory Course — Fachbereich Mathematik
Winter Semester 2008/2009 — Dennis Frisch

Hints to Exercise, Unit 7

1. We apply the definition of a limit of a function and show that f has a limit in $a \in \mathbb{R}$:

   Let $(a_n)_{n\in\mathbb{N}}$ be a sequence with limit a. Then

   $$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} a_n = a = f(a).$$

   shows that f has the limit a in a.

   This shows that f is continuous in a. As a was an arbitrary real number this shows that f is continuous in $\mathbb{R}$.

2. One has to check that the value $cx + 1$ and $cx^2 - 1$ coincide for $x = 3$ and that each of the two branches converges to it (which is obvious as these are continuous functions). $c = \frac{1}{3}$.

3. Should be straight forward: Apply the Intermediate Value Theorem to the function $f(x) = x^2$ on the interval $[1, 2]$.

4. Also a simple application of the IVT.

5. We look at

   $$|f(x_0) - f(x)| = |x_0^2 - x^2| = |(x_0 - x)(x_0 + x)| = |x_0 - x|\,|x_0 + x| \overset{hint}{\leq} (2|x_0| + 1)|x_0 - x|.$$

   If this is less than ε, then we can choose $\delta := \frac{\varepsilon}{2|x_0|+1}$. Since the last estimate is only true is $|x_0 - x| \leq 1$, we have to choose

   $$\delta := \min\left\{1, \frac{\varepsilon}{2|x_0| + 1}\right\}.$$

6. All statements except the last one followed by our theroem about algebra of sequences. For example we show that $f + g$ is continous:

   The continuity of f implies, that for $x_n \to x_0$ also $f(x_n) \to f(x_0)$. The same for g by continuity of g. this yields to

   $$\lim_{n\to\infty}(f+g)(x_n) = \lim_{n\to\infty} f(x_n)+g(x_n) = \lim_{n\to\infty} f(x_n)+\lim_{n\to\infty} g(x_n) = f(x_0)+g(x_0) = (f+g)(x_0).$$

   For the continuity of $f \circ g$, we have to do it in another way. So if we take a convergent sequence $x_n \to x_0$, then continuity of g implies, that

   $$g(x_n) \longrightarrow g(x_0).$$

   So $g(x_n)$ is a convergent sequence. Then the continuity of f implies, that

   $$f(g(x_n)) \longrightarrow f(g(x_0)).$$

   All together yields to

   $$\lim_{n\to\infty}(f\circ g)(x_n) = \lim_{n\to\infty} f(g(x_n)) \overset{f cont.}{=} f\left(\lim_{n\to\infty} g(x_n)\right) \overset{g cont.}{=} f\left(g\left(\lim_{n\to\infty} x_n\right)\right) = f(g(x_0)) = (f\circ g)(x_0).$$