Mathematics with Computer Science

Technische Universität Darmstadt
Fachbereich Mathematik
Dennis Frisch

Introductory Course
Winter Semester 2008/2009

Hints to Exercise, Unit 6

1. (i) We estimate

$$\sum_{n=1}^{\infty} \frac{n+1}{n^3} \leq \sum_{n=1}^{\infty} \frac{n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

So we've found a majorant and so the series is convergent.

(ii) It is easy to see, that $n! > 3^n$ for $n > 7$. So

$$\lim_{n\to\infty} \frac{n!}{3^n} \neq 0.$$

Therefore the series can't converge.

(iii) We estimate

$$\frac{n!}{n^n} = \frac{n \cdot (n-1) \cdots 2 \cdot 1}{n \cdot n \cdot \cdots \cdot n \cdot n} \leq \frac{n \cdot n \cdots n \cdot 2 \cdot 1}{n \cdot n \cdot \cdots \cdot n \cdot n} = \frac{2}{n^2}.$$

So we get

$$\sum_{n=1}^{\infty} \frac{n!}{n^n} \leq \sum_{n=1}^{\infty} \frac{2}{n^2} = 2 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

2. a) We have a look at the equation

$$\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}.$$

Multiplying $(1-x)$ (for $x \neq 1$) yields to

$$(1-x)\sum_{k=0}^{n} x^k = 1 + x^{n+1}.$$

If we look at the left side, we get

$$\begin{aligned}
(1-x)\sum_{k=0}^{n} x^k &= \sum_{k=0}^{\infty} x^k(1-x) \\
&= \sum_{k=0}^{\infty} x^k - x^{k+1} \\
&= \underbrace{1-x^1}_{k=0} + \underbrace{x-x^2}_{k=1} + \underbrace{x^2-x^3}_{k=3} + \ldots + \underbrace{x^{n-1}-x^n}_{k=n-1} + \underbrace{x^n-x^{n+1}}_{k=n} \\
&= 1-x^{n-1}.
\end{aligned}$$

b)

$$\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{1-\frac{2}{3}} = 3.$$

3. Assume $\sum a_n$ is convergent, that means that the sequence $(s_n)_{n\in\mathbb{N}}$ of partial sums is convergent. Because $a_n \geq 0$ (s_n) is monotone increasing. By Theorem 6.2.2 the sequence (s_n) of partial sums is bounded by C. Then we conclude

$$\sum_{k=0}^{n} x_k \leq \sum_{k=0}^{n} a_k \leq C.$$

Since $x_n \geq 0$ too, we had also that the sequence (t_n) of partial sums of $\sum x_n$ is monotone increasing and bounded. That means, that $\sum x_n$ is convergent too, which is a contradiction.

4. (i) We estimate

$$\sum_{n=0}^{\infty} \frac{2^n}{3^n+8} \leq \sum_{n=0}^{\infty} \frac{2^n}{3^n} = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n.$$

By Problem 2.b) this is equals to 3 and therefor the series is convergent.

(ii) We estimate

$$\sum_{n=2}^{\infty} \frac{1}{n-1} \geq \sum_{n=2}^{\infty} \frac{1}{n}.$$

So the series is not convergent since $\sum \frac{1}{n}$ is a minorant according to problem 3.

(iii) We estimate

$$\sum_{n=0}^{\infty} \frac{1}{n^2-n+1} \leq 1+1+\sum_{n=2}^{\infty} \frac{1}{n^2-\frac{1}{2}n^2} = 2+2\sum_{n=2}^{\infty} \frac{1}{n^2}.$$

Again we had the majorant $\sum \frac{1}{n^2}$ and we can conclude that the series is convergent.