Mathematics with Computer Science

Technische Universität Darmstadt
Fachbereich Mathematik
Dennis Frisch

Introductory Course
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Hints to Exercises 5

1. (a) Standard argument.
   (b) Solve the inequality $|\frac{2n-3}{5n+7} - \frac{2}{5}| \leq \frac{1}{10}$ for n.

2. (a) Shown (for example by induction) that $2^n > n$. This implies $\frac{1}{2^n} < \frac{1}{n}$. The limit is 0.
   (b) Set
$$a_n = \sum_{k=1}^{n} \frac{1}{2^k}.$$
   Then
$$a_n = 1 - \frac{1}{2^n}.$$
   Therefore, the limit is 1.
   Consider the interval from 0 to 1 on the real line. Adding up the sum above means that we add half of the interval. Then we add half of the rest to that ($\frac{1}{4}$). Then again half of the remainder ($\frac{1}{8}$) and so on. This illustrates that the limit is 1.

3. The first terms of an infinite sequence are $1, 3, 7, 15, 31, 63$.
   (a) E.g.: $a_1 = 1$, $a_{n+1} = 2a_n + 1$.
   (b) $a_n = 2^n - 1$

4. $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2a_n}$. The limit is 2.

5. Determine the limit (if it exists) of
$$\lim_{n\to\infty} a_n = 0, \qquad \lim_{n\to\infty} b_n = -3, \qquad \lim_{n\to\infty} c_n = \frac{2}{3},$$
$$\lim_{n\to\infty} d_n = 0, \qquad \lim_{n\to\infty} e_n \text{ does not exist}$$

6. (a)
$$a_n = \sqrt{n^2+1} - n = \frac{(\sqrt{n^2+1}-n)(\sqrt{n^2+1}+n)}{\sqrt{n^2+1}+n} = \frac{n^2+1-n^2}{\sqrt{n^2+1}+n} = \frac{1}{\sqrt{n^2+1}+n} \to 0$$
   (b)
$$b_n = \frac{n}{\sqrt{n^2+1}+n} = \frac{1}{\sqrt{(1+\frac{1}{n^2})}+1} \to \frac{1}{2}$$
   (c)
$$c_n = \frac{n^2}{\sqrt{(n^2+1)}+n} = \frac{n}{\sqrt{(1+\frac{1}{n^2})}+1} > \frac{n}{\sqrt{1}+1} > n$$