Mathematics with Computer Science

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Introductory Course — Fachbereich Mathematik
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Hints to Exercises 3

1. We compute

$$b^3 = (a + \frac{1}{a})^3 = a^3 + \frac{1}{a^3} + 3(a + \frac{1}{a}) = a^3 + \frac{1}{a^3} + 3b.$$

The statement follows.

2. We show that $\frac{3x-4}{2x+4} \leq -1$ implies $x \leq 0$:

$$\frac{3x-4}{2x+4} \leq -1 \Rightarrow 3x - 4 \leq -2x - 4 \Rightarrow 5x \leq 0 \Rightarrow x \leq 0.$$

3. (a) If n is even, it is divisible by 2. Then n^2 is also divisible by 2.

   If n is odd, it is of the form $2k - 1$ for a natural number k. Then $n^2 = 4k^2 - 4k + 1 = 2(2k^2 - 2k) + 1$ is obviously odd.

   (b) We follow the proof that $x^2 = 2$ does not have a rational solution. Set $x = \frac{a}{b}$ with a and b having no common factor. The key observation is that the equation $a^2 = 6b^2$ implies that a is even. Then we get $a = 4c$ for some c and $2c^2 = 3b^2$. This equation implies that b is even which is a contradiction.

   (c) If $a = 0$, then we have to consider $r = b\sqrt{2}$. If r is rational, then $\sqrt{2} = \frac{r}{b}$ is also rational.

   Assume $a \neq 0$. Then $r = (a + b\sqrt{2})^2 = a^2 + 2ab\sqrt{2} + 2b^2$ and $\sqrt{2} = \frac{r-a^2-2b^2}{2ab}$. If r is rational, then $\sqrt{2}$ is also rational.

   (d) Use the proof that shows that $x^2 = 2$ does not have a rational solution.

4. Induction start: For $n = 1$ we get

$$\sum_{k=1}^{1} k^2 = 1^2 = 1 = \frac{1(1+2)(2 \cdot 1 + 1)}{6}.$$

Induction step: Assume we know the claim to hold for some value of n, i.e.

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Then

$$\begin{aligned}\sum_{k=1}^{n+1} k^2 &= \sum_{k=1}^{n} k^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} = \frac{(n+1)\left(n(2n+1) + 6(n+1)\right)}{6} \\ &= \frac{(n+1)\left(2n^2 + 7n + 6\right)}{6} = \frac{(n+1)(n+2)(2n+3)}{6} \\ &= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}.\end{aligned}$$

5. Both propositions can be shown by induction.

   (a) *Induction start:* For $n = 4$ is

   $$2 \cdot 4 + 1 = 9 < 16 = 4^2.$$

   Induktion step: We assume, that the claim holds for a given $n \in \mathbb{N}$. Then we get

   $$\begin{aligned}2(n+1) + 1 &= 2n + 1 + 2 \\ &\overset{\text{Assumption}}{<} n^2 + 2 \overset{n>3}{<} n^2 + 2n < n^2 + 2n + 1 \\ &= (n+1)^2.\end{aligned}$$

   (b) *Induction start:* For $n = 4$ is

   $$4^2 = 16 \leq 16 = 2^4.$$

   Induktion step: We assume, that the claim holds for a given $n \in \mathbb{N}$. Then we get

   $$\begin{aligned}(n+1)^2 &= n^2 + 2n + 1 \overset{\text{a)}}{<} n^2 + n^2 = 2n^2 \\ &\overset{\text{Assumption}}{\leq} 2 \cdot 2^n \\ &= 2^{n+1}.\end{aligned}$$

6. The error is in the last step. The equation is formulated over the complex numbers and the equation

   $$x^3 = 1$$

has three different solutions over the complex numbers. So substituting only one solution isn't an equivalence transformation.

There has to be another mistake, but I can't exactly find out at the moment. Sorry for that, I try to find it out.