# Mathematics with Computer Science

Technische Universität Darmstadt
Fachbereich Mathematik
Dennis Frisch

Introductory Course
Winter Semester 2008/2009

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## Hints to exercises 1

1. We show this by inspecting the multiples of 7 and observing that $100$ is not among them. Therefore, 7 is not a divisor of $100$:

$$7, 14, 21, 35, 42, 49, 56, 63, 70, 77, 84, 91, 98, 105, \ldots$$

2. The divisors of 12 are $\{1, 2, 3, 4, 6, 12\}$.

   Any divisor of 12 is smaller than or equal to $12$. We need to show that $5$, $7$, $8$, $9$, $10$ and $11$ are not divisors of $12$.

   We show the assertion for $5$: The multplies of $5$ are $5, 10, 15, \ldots$ and we see that $12$ is not among the multiples of $5$. Therefore, $5$ is not a divisor of $12$. It is easy to treat the other non-divisors of $12$ in the same way.

   The divisors of $140$ are $\{1, 2, 4, 5, 7, 10, 14, 20, 28, 35, 70, 140\}$.

   The divisors of $1001$ are $\{1, 7, 11, 13, 77, 91, 143, 1001\}$.

3. The following table is the sieve after the multiples of 7 have been crossed out.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | ~~4~~ | 5 | ~~6~~ | 7 | ~~8~~ | ~~9~~ | ~~10~~ |
| 11 | ~~12~~ | 13 | ~~14~~ | ~~15~~ | ~~16~~ | 17 | ~~18~~ | 19 | ~~20~~ |
| ~~21~~ | ~~22~~ | 23 | ~~24~~ | ~~25~~ | ~~26~~ | ~~27~~ | ~~28~~ | 29 | ~~30~~ |
| 31 | ~~32~~ | ~~33~~ | ~~34~~ | ~~35~~ | ~~36~~ | 37 | ~~38~~ | ~~39~~ | ~~40~~ |
| 41 | ~~42~~ | 43 | ~~44~~ | ~~45~~ | ~~46~~ | 47 | ~~48~~ | ~~49~~ | ~~50~~ |

   We see that all remaining numbers are primes and can stop the process.

   Observe that when you cross out the multiples of a number $n$, then the first multiple of $n$ that is not crossed out yet is $n^2$. Why is that the case?

4. We write $n = 1 \cdot n$. We compare this with the definition of divisibility and see that there is a number $d$ (which is $n$ in this case) such that $n = d \cdot 1$. We also see that there is another number $d$ (which is $1$ in this case) such that $n = d \cdot n$.

5. If $p$ is a divisor of $q$, then there is a natural number $d_1$ such that $q = d_1 \cdot p$. Likewise, there is a natural number $d_2$ such that $r = d_2 \cdot q$. Substituting the first equation into the second gives: $r = d_2 d_1 \cdot p$ and we see that $p$ divides $r$.

6. Let $d \geq \sqrt{n}$. As $d$ divides $n$, there is an $m$ such that $dm = n$. Then

$$d \cdot m = n = \sqrt{n} \cdot \sqrt{n} \leq d \cdot \sqrt{n}.$$

   Cancelling $d$ on both ends gives $m \leq \sqrt{n}$.

   If we have checked that a number $n$ is not divisible by any of the natural numbers (different from $1$) smaller than or equal to $\sqrt{n}$, then it cannot be divisible by any of the numbers strictly between $1$ and $n$. This means that we can stop the method of trial division at $\sqrt{n}$.

   This coincides with the observation that the Eratosthenes' Sieve in Problem $3$ could be stopped when the multiples of the prime 7 were crossed out. The next prime, 11, is larger than $\sqrt{50}$.

7. If $m = dq$ and $n = dr$, then $m + n = dq + dr = d(q + r)$. Hence $d$ divides $m + n$. Write $mn = dq \cdot dr = d^2 \cdot qr$.

8.

$$\begin{aligned}
\sum_{i=0}^{3} i &= 0 + 1 + 2 + 3 = 6 \\
\sum_{j=1}^{3} 3^j &= 3 + 9 + 27 = 39 \\
\sum_{k=1}^{2} 2 + j &= 2 + j + 2 + j = 4 + 2j
\end{aligned}$$

9.

$$\begin{aligned}
1 + 3 + 5 + 7 + 9 &= \sum_{i=0}^{4} 2i + 1 \\
2 + 4 + 6 + 8 + 10 &= \sum_{i=1}^{5} 2i \\
2 + 4 + 8 + 16 &= \sum_{i=1}^{4} 2^i
\end{aligned}$$

10. Direct computation:

$$\begin{aligned}
&\frac{n!}{(n-(k-1))!(k-1)!} + \frac{n!}{(n-k)!k!} \\
&= \frac{n!}{(n-k+1)!(k-1)!} + \frac{n!}{(n-k)!k!} \\
&= \frac{n! \cdot k}{(n-k+1))!k!} + \frac{n!(n-k+1)}{(n-k+1)!k!} \\
&= \frac{n! \cdot k + n!(n-k+1)}{(n-k+1))!k!} \\
&= \frac{(n+1)!}{(n+1-k)!k!}
\end{aligned}$$

11. From the binomial formula: Substitute $x = 1$ and $y = 1$ into $(x + y)^n$.

From the binomial formula: Substitute $x = 1$ and $y = -1$ into $(x + y)^n$.

12. (a) We distinguish two cases:

    i. $a \geq 0$. Then $|a| = a$ and $-a \leq 0 \leq a$. Thus

$$|a| \leq c \Leftrightarrow a \leq c \Leftrightarrow a \leq c \text{ and } -c \leq -a \leq a \Leftrightarrow -c \leq a \leq c.$$

    ii. $a < 0$. Then $|a| = -a$ and $a < 0 < -a$. Thus

$$|a| \leq c \Leftrightarrow -a \leq c \Leftrightarrow -a \leq c \text{ and } -c \leq a < -a \Leftrightarrow -c \leq a \leq c.$$

(b) Use (a) with $c = |a|$ and note that $|a| \leq |a|$ is always true.

(c) By (b) we have $a \leq |a|$ and $b \leq |b|$ thus $a + b \leq |a| + |b|$. Likewise we have $-|a| \leq a$ and $-|b| \leq b$ thus $-|a| - |b| \leq a + b$. Combining this yields $-(|a| + |b|) \leq a + b \leq |a| + |b|$ which by (a) is equivalent to $|a + b| \leq |a| + |b|$

(d) Set $c := a - b$. Bei (c) we have $|a| = |b + c| \leq |b| + |c| = |b| + |a - b|$.

(e) By (a) $|x - a| \leq \varepsilon$ is the same as saying $-\varepsilon \leq x - a \leq \varepsilon$. Adding $a$ to this inequality yields the desired statement.

Geometrically, when looking at the number line, this indicates all real numbers (points on the number line) that are at distance at most $\varepsilon$ from $a$. So, the $x$ satisfying it form a line segment, the closed interval $[a - \varepsilon, a + \varepsilon]$.

(f) To find the solutions of $|4 - 3x| > 2x + 10$ we distinguish between two cases:

    i. $4 - 3x \geq 0$: Equivalentely, $x \leq \frac{4}{3}$. Then the above inequality becomes

$$4 - 3x > 2x + 10 \Leftrightarrow -6 > 5x \Leftrightarrow x < -\frac{6}{5} \leq \frac{4}{3}.$$

    ii. $4 - 3x < 0$: Equivalentely, $x > \frac{4}{3}$. Then the above inequality becomes

$$-(4 - 3x) > 2x + 10 \Leftrightarrow x > 14 > \frac{4}{3}.$$

Hence the solutions are those real number which are either greater than $14$ or less than $-\frac{6}{5}$. The second problem can be solved in a similar fashion, showing that $\frac{10}{3} \leq x \leq 10$ is the set of solutions there.