2 Discrete-Time Martingales

Let $(\Omega, \mathfrak{A}, P)$ be a fixed probability space. We will call a sequence $X = (X_n)_{n\in\mathbb{N}_0}$ of real–valued r.v. a *stochastic process* over the natural numbers.

Definition 1. A sequence $\mathfrak{F} = (\mathfrak{F}_n)_{n\in\mathbb{N}_0}$ of σ-algebras $\mathfrak{F}_n \subset \mathfrak{F}$ is called a *filtration (in $(\Omega, \mathfrak{A})$)* iff

$$\forall\, n \in \mathbb{N}_0 : \ \mathfrak{A}_n \subset \mathfrak{A}_{n+1} \ .$$

Example 1. For a stochastic process $X = (X_n)_{n\in\mathbb{N}_0}$, the σ–algebras

$$\mathfrak{F}_n^X := \sigma(\{X_0, \ldots, X_n\}), \qquad n \in \mathbb{N}_0, \tag{1}$$

build a filtration $\mathfrak{F}^X$, the *canonical filtration* of X. It is the smallest filtration $\mathfrak{F}$ such that X_n is always $\mathfrak{F}_n$–measurable. Further, some $Y : \Omega \to \mathbb{R}$ is $\mathfrak{F}_n^X$–measurable iff there is $g : \mathbb{R}^{n+1} \to \mathbb{R}$ measurable such that $Y = g(X_0, \ldots, X_n)$, see Theorem II.2.8 und Corollary II.3.1.(i).

A filtration shall encode which part of the whole 'randomness' contained in $(\Omega, \mathfrak{A}, P)$ has unfolded up to time n; in the case of a canonical filtration $\mathfrak{F}^X$, the 'randomness up to time n' comes solely from the result of drawing $X_0, \ldots, X_n$ randomly.

Definition 2. A stochastic process X is called a *martingale (with respect to $\mathfrak{F}$, or $\mathfrak{F}$–martingale)* iff $X_n \in \mathfrak{L}^1$ for all $n \in \mathbb{N}_0$ and

$$\forall\, {}^{n,m\in\mathbb{N}_0}_{n<n} \ : \ \mathrm{E}(X_m \,|\, \mathfrak{F}_n) = X_n.$$

Theorem 1.3 allows to interprete: For a martingale X, the best predictor for X_m, given all knowledge available at time $n < m$, is just X_n.

Remark 1. For an X $\mathfrak{F}$–martingale and $n < m$ we have

$$\mathrm{E}(X_m) = \mathrm{E}(\mathrm{E}(X_m \,|\, \mathfrak{A}_n)) = \mathrm{E}(X_n).$$

But of course, this is not a sufficient condition for the martingale property.

Remark 2. By the Towering Lemma 1.3 and simple induction it follows that X is an $\mathfrak{F}$–martingale iff

$$\forall\, n \in \mathbb{N}_0 : \quad \mathrm{E}(X_{n+1} \,|\, \mathfrak{A}_n) = X_n.$$

Example 2. Let $(Y_i)_{i\in\mathbb{N}}$ be independent with constant expectation $\mathrm{E}(Y_i) = a$. Set $\mathfrak{F}_0 = \{\emptyset, \Omega\}$ and $\mathfrak{F}_n = \sigma(\{Y_1, \ldots, Y_n\})$ for $n \geq 1$. We define $X_0 = 0$ and

$$X_n = \sum_{i=1}^{n} Y_i, \qquad n \in \mathbb{N}.$$

It is easy to see that $\mathfrak{F} = \mathfrak{F}^X$ and

$$\mathrm{E}(X_{n+1} \,|\, \mathfrak{F}_n) = \mathrm{E}(X_n \,|\, \mathfrak{F}_n) + \mathrm{E}(Y_{n+1} \,|\, \mathfrak{F}_n) = X_n + \mathrm{E}(Y_{n+1}) = X_n + a.$$

Hence X is an $\mathfrak{F}$–martingale iff $a = 0$. As special cases we have the case of Y_i being i.i.d., i.e., a random walk.
Let us interprete now Y_i as the gain/loss of some game in round i by a fixed betting amount (say, 1); then X_n is the cumulative gain/loss of n such game rounds with betting amount 1. Then X_n is a martingale iff $a = 0$, i.e., if the game is 'fair'. A question raised time and again through the millennia is: If one cleverly (in particular, depending on the results of the previous rounds) chooses

(i) the amount to bet in the i–th round,

(ii) a time when to stop and go home,

can one 'beat the system', i.e., get more on the average than with X_n?

Example 3. The *Cox-Ross-Rubinstein model* for stock prices X_n at discrete times $n \in \mathbb{N}_0$. Choose some real parameters

$$X_0 > 0, \quad 0 < p < 1, \quad 0 < d < u,$$

and let $(Y_i)_{i\in\mathbb{N}}$ be i.i.d. with

$$P(\{Y_i = u\}) = p = 1 - P(\{Y_i = d\}).$$

Define now

$$X_n = X_0 \cdot \prod_{i=1}^{n} Y_i$$

and consider $\mathfrak{F} = \mathfrak{F}^X$. By Lemmas 1.2 and 1.4 we have

$$\mathrm{E}(X_m \,|\, \mathfrak{F}_n) = X_n \cdot \mathrm{E}\left(\prod_{\ell=n+1}^{m} Y_\ell\right) = X_n \cdot \mathrm{E}(Y_1)^{m-n}.$$

Hence,

$$\widetilde{X} \text{ martingale} \qquad \Leftrightarrow \qquad \mathrm{E}(Y_1) = 1,$$

and in terms of p,

$$\widetilde{X} \text{ Martingal} \qquad \Leftrightarrow \qquad d < 1 < u \ \wedge \ p = \frac{1-d}{u-d}.$$

The same question as in example 2 occurs: Are there clever trading strategies which allow to make a profit on the average? (Further, is it a good idea to model the stock prices as a martingale?)

Let in the following be fixed:

(i) An $\mathfrak{F}$–martingale $X = (X_n)_{n\in\mathbb{N}_0}$,

(ii) a stochastic process $H = (H_n)_{n\in\mathbb{N}_0}$ such that

$$\forall\, n \in \mathbb{N}_0 : \ H_n \ \mathfrak{F}_n\text{-measurable} \wedge \ H_n \cdot (X_{n+1} - X_n) \in \mathfrak{L}^1.$$

Definition 3. The stochastic process $Z = (Z_n)_{n\in\mathbb{N}_0}$ with $Z_0 = 0$ and

$$Z_n = \sum_{i=0}^{n-1} H_i \cdot (X_{i+1} - X_i), \qquad n \geq 1,$$

is called *martingale transformation of* X *by* H. Shorthand: $Z = H \bullet X$.

(This is a discrete version of a stochastic integral.)

Example 4. In Example 2: H_n is the amount one wagers in the $(n+1)$st (!) game or the amount of stock one buys at time n and sells at time $n+1$; it can be chosen cleverly, but using only knowledge obtainable at time n; this is modeled by the assumption that H_n is $\mathfrak{F}_n$–measurable. $Z = H \bullet X$ then is the cumulative gain at time n when the strategy H was used.

Theorem 1 (No way to beat the system). $Z = H \bullet X$ is an $\mathfrak{F}$–martingale.

This result says that no matter how clever (or stupid) I choose my strategy, I cannot escape the martingale setting.

Proof. Obviously, Z_n is $\mathfrak{F}_n$–measurable and in $\mathfrak{L}^1$; further,

$$\mathrm{E}(Z_{n+1} \,|\, \mathfrak{A}_n) = Z_n + \mathrm{E}(H_n \cdot (X_{n+1} - X_n) \,|\, \mathfrak{A}_n),$$

and by the Towering Lemma 1.2,

$$\mathrm{E}(H_n \cdot (X_{n+1} - X_n) \,|\, \mathfrak{A}_n) = H_n \cdot \mathrm{E}((X_{n+1} - X_n) \,|\, \mathfrak{A}_n) = 0.$$

□