# Chapter IV

# Limit Theorems

**Motivation:** Given a sequence of random variables $X_n$, $n \in \mathbb{N}$, on a probability space $(\Omega, \mathfrak{A}, P)$, put

$$S_n = \sum_{i=1}^{n} X_i, \qquad n \in \mathbb{N}.$$

For instance, $S_n$ might be the cumulative gain after $n$ trials or (one of the coordinates of) the position of a particle after $n$ collisions.

*Question:* Convergence of $S_n/a_n$ for suitable weights $0 < a_n \uparrow \infty$ in a suitable sense? Particular case: $\boxed{a_n = n}$.

## 1 Zero-One Laws

Zero–One Laws explain when and in which sense a 'limit' of random events is non-random. In the language of random variables, it explains when and in which sense limits of random variables will degenerate, i.e., become constant almost surely.

**Definition 1.** For $\sigma$-algebras $\mathfrak{A}_n \subset \mathfrak{A}$, $n \in \mathbb{N}$, the corresponding *tail $\sigma$-algebra* is

$$\mathfrak{A}_\infty = \bigcap_{n \in \mathbb{N}} \sigma\Big( \bigcup_{m \geq n} \mathfrak{A}_m \Big),$$

and $A \in \mathfrak{A}_\infty$ is called a *tail (terminal) event*.

It is useful to think of $\mathfrak{A}_n$ as a $\sigma$–algebra containing information about 'random events' happening at some 'time' $n$. Terminal events can then be considered as limits of events which are 'decided in the distant future'. Analogously, $\mathfrak{A}_\infty$–measurable r.v. are those r.v. whose values are 'determined in the distant future'.

**Example 1.** Let $\mathfrak{A}_n = \sigma(X_n)$. Then

$$\mathfrak{A}_\infty = \bigcap_{n \in \mathbb{N}} \sigma(\{X_m : m \geq n\})\ .$$

For instance,

$$\{(S_n)_{n\in\mathbb{N}} \text{ converges}\},\ \{(S_n/a_n)_{n\in\mathbb{N}} \text{ converges}\} \in \mathfrak{A}_\infty,$$

and the function $\liminf_{n\to\infty} S_n/a_n$ is $\mathfrak{A}_\infty$-$\overline{\mathfrak{B}}$-measurable. However, $S_n$ as well as $\liminf_{n\to\infty} S_n$ are not $\mathfrak{A}_\infty$-$\overline{\mathfrak{B}}$-measurable, in general (why?). Analogously for the $\limsup$'s.

**Theorem 1 (Kolmogorov's Zero-One Law).** Let $(\mathfrak{A}_n)_{n\in\mathbb{N}}$ be an *independent* sequence of $\sigma$-algebras $\mathfrak{A}_n \subset \mathfrak{A}$. Then

$$\forall\, A \in \mathfrak{A}_\infty :\ P(A) \in \{0,1\}\,.$$

This theorem says that terminal events of independent $\sigma$–algebras are actually non-random.

*Proof.* Since, for any $n$, $\mathfrak{A}_\infty \subset \sigma(\mathfrak{A}_{n+1} \cup \dots)$, the family $(\mathfrak{A}_\infty, \mathfrak{A}_1, \mathfrak{A}_2, \dots)$ is independent. By Corollary III.5.1, $\mathfrak{A}_\infty$ is independent from $\sigma(\bigcup_n \mathfrak{A}_n)$. On the other hand,

$$\mathfrak{A}_\infty \subset \sigma\Bigg(\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n\Bigg)\,.$$

Thus, $\mathfrak{A}_\infty$ is independent of itself; in particular, for $A \in \mathfrak{A}_\infty$, we have $P(A) = P(A \cap A) = P(A)^2$. But this is only possible if $P(A) \in \{0,1\}$. □

**Corollary 1.** Let $X \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A}_\infty)$. Under the assumptions of Theorem 1, $X$ is constant $P$-a.s.

*Proof.* For each $c \in \overline{\mathbb{R}}$,

$$P(X = c) = P(\underbrace{X^{-1}(\{c\})}_{\in \mathfrak{A}_\infty}) \in \{0,1\}\,.$$

□

**Remark 1.** Assume that $(X_n)_{n\in\mathbb{N}}$ is independent. Then

$$P(\{(S_n)_{n\in\mathbb{N}} \text{ converges}\}),\ P(\{(S_n/a_n)_{n\in\mathbb{N}} \text{ converges}\}) \in \{0,1\}.$$

In case of convergence $P$-a.s., $\lim_{n\to\infty} S_n/a_n$ is constant $P$-a.s.

We know now that terminal events either happen 'essentially always' or 'essentially never'. But how to decide which case holds? A very powerful and surprisingly universal tool is Borell–Cantelli's Lemma, which we will develop in the sequel.

**Definition 2.** Let $A_n \in \mathfrak{A}$ for $n \in \mathbb{N}$. Then

$$\varliminf_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} A_m, \qquad \varlimsup_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} A_m.$$

A more intuitive reformulation is as follows: $\omega \in \underline{\lim}_n A_n$ iff $\omega \in A_m$ holds for all $m \geq m_0(\omega)$; in other words, $\underline{\lim}_n A_n$ happens iff 'eventually all' $A_m$ happen. $\omega \in \overline{\lim}_n A_n$ iff $\omega \in A_m$ for infinitely many $A_m$; in other words, $\overline{\lim}_n A_n$ happens iff 'infinitely often' $A_m$ happens.

**Remark 2.**

(i) $\left(\underline{\lim}_{n\to\infty} A_n\right)^c = \overline{\lim}_{n\to\infty} A_n^c$.

(ii) $P\left(\underline{\lim}_{n\to\infty} A_n\right) \leq \underline{\lim}_{n\to\infty} P(A_n) \leq \overline{\lim}_{n\to\infty} P(A_n) \leq P\left(\overline{\lim}_{n\to\infty} A_n\right)$.

(iii) If $(A_n)_{n\in\mathbb{N}}$ is independent, then $P\left(\overline{\lim}_{n\to\infty} A_n\right) \in \{0,1\}$ (*Borel's Zero-One Law*).

Proof: Übung 11.1, 11.2.

**Theorem 2 (Borel-Cantelli Lemma).** Let $A = \overline{\lim}_{n\to\infty} A_n$ with $A_n \in \mathfrak{A}$.

(i) If $\sum_{n=1}^{\infty} P(A_n) < \infty$ then $P(A) = 0$.

(ii) If $\sum_{n=1}^{\infty} P(A_n) = \infty$ and $(A_n)_{n\in\mathbb{N}}$ are independent, then $P(A) = 1$.

*Proof.* Ad (i): For all $n$,

$$P(A) \leq P\left(\bigcup_{m\geq n} A_m\right) \leq \sum_{m=n}^{\infty} P(A_m) \overset{n\to\infty}{\longrightarrow} 0.$$

Ad (ii): We have

$$P(A^c) = P(\underline{\lim}_{n\to\infty} A_n^c) \leq \sum_{n=1}^{\infty} P\left(\bigcap_{m\geq n} A_m^c\right) = \sum_{n=1}^{\infty} \inf_{l\geq n} \prod_{m=n}^{\ell} (1 - P(A_m)).$$

Use $1 - x \leq \exp(-x)$ for $x \geq 0$ to obtain

$$\prod_{m=n}^{\ell} (1 - P(A_m)) \leq \prod_{m=n}^{\ell} \exp(-P(A_m)) = \exp\left(-\sum_{m=n}^{\ell} P(A_m)\right).$$

By assumption, the right-hand side tends to zero as $\ell$ tends to $\infty$. Thus $P(A^c) = 0$. □

**Example 2.** A fair coin is tossed an infinite number of times. Determine the probability that 0 occurs twice in a row infinitely often. Model: $(X_n)_{n\in\mathbb{N}}$ is independent and

$$P(\{X_n = 0\}) = P(\{X_n = 1\}) = 1/2, \qquad n \in \mathbb{N}.$$

Put

$$A_n = \{X_n = X_{n+1} = 0\}.$$

Then $(A_{2n})_{n\in\mathbb{N}}$ is independent and $P(A_{2n}) = 1/4$. Thus $P(\overline{\lim}_{n\to\infty} A_n) = 1$.

**Example 3.** Let $a_n \in [0,1]$ and $(X_n)_{n\in\mathbb{N}}$ a sequence of independent r.v., where $X_n$ is uniformly distributed on $[0,1]$. Consider the event

$$\Big\{X_n \leq a_n \text{ infinitely often}\Big\} = \overline{\lim_n} \underbrace{\{X_n \leq a_n\}}_{=:A_n} .$$

Then $P(A_n) = a_n$; hence

$$\Big\{X_n \leq a_n \text{ infinitely often}\Big\} = \begin{cases} 0, & \sum_n a_n < \infty , \\ 1, & \sum_n a_n = \infty . \end{cases}$$

In particular, a sequence of numbers 'drawn randomly from $[0,1]$' contains almost surely a subsequence $n_k$ tending to zero faster than $1/n_k$, but almost surely no subsequence $n_k$ tending to zero faster than $1/n_k^2$.