4 Uniform Integrability

In the sequel: X_n, X random variables on a common probability space $(\Omega, \mathfrak{A}, P)$.

Definition 1. $(X_n)_{n\in\mathbb{N}}$ *uniformly integrable (u.i.)* if

$$\lim_{\alpha\to\infty} \sup_{n\in\mathbb{N}} \int_{\{|X_n|\geq\alpha\}} |X_n| \, dP = 0.$$

Remark 1.

(i) $(X_n)_{n\in\mathbb{N}}$ u.i. $\Rightarrow$ $(\forall n \in \mathbb{N}: \; X_n \in \mathfrak{L}^1) \wedge \sup_{n\in\mathbb{N}} \|X_n\|_1 < \infty$.

(ii) $[\exists\, Y \in \mathfrak{L}^1 \; \forall n \in \mathbb{N}: \; |X_n| \leq Y] \Rightarrow (X_n)_{n\in\mathbb{N}}$ u.i.

(iii) If $[\exists\, p > 1 \;\; (\forall n \in \mathbb{N}: X_n \in \mathfrak{L}^p) \wedge \sup_{n\in\mathbb{N}} \|X_n\|_p < \infty]$, then

$$\int_{\{|X_n|\geq\alpha\}} |X_n| \, dP = 1/\alpha^{p-1} \cdot \int_{\{|X_n|\geq\alpha\}} \alpha^{p-1}|X_n| \, dP \leq 1/\alpha^{p-1} \cdot \|X_n\|_p^p$$

and hence $(X_n)_{n\in\mathbb{N}}$ u.i..

Example 1. For the uniform distribution P on $[0,1]$ and

$$X_n = n \cdot 1_{[0,1/n]}$$

we have $X_n \in \mathfrak{L}^1$ and $\|X_n\|_1 = 1$, but for any $\alpha > 0$ and $n \geq \alpha$

$$\int_{\{|X_n|\geq\alpha\}} |X_n| \, dP = n \cdot P([0,1/n]) = 1,$$

so that $(X_n)_{n\in\mathbb{N}}$ is not u.i.

Lemma 1. $(X_n)_{n\in\mathbb{N}}$ u.i. iff $\sup_{n\in\mathbb{N}} \mathrm{E}(|X_n|) < \infty$ and

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall A \in \mathfrak{A}: \quad P(A) < \delta \; \Rightarrow \; \sup_{n\in\mathbb{N}} \int_A |X_n| \, dP < \varepsilon. \tag{1}$$

Proof. '$\Rightarrow$': $\sup_{n\in\mathbb{N}} \mathrm{E}(|X_n|) < \infty$ by Remark 1.(i). Moreover, for $A \in \mathfrak{A}$ we have

$$\begin{aligned}\int_A |X_n| \, dP &= \int_{A\cap\{|X_n|\geq\alpha\}} |X_n| \, dP + \int_{A\cap\{|X_n|<\alpha\}} |X_n| \, dP \\ &\leq \int_{\{|X_n|\geq\alpha\}} |X_n| \, dP + \alpha \cdot P(A).\end{aligned}$$

For $\varepsilon > 0$ take $\alpha > 0$ with

$$\sup_{n\in\mathbb{N}} \int_{\{|X_n|\geq\alpha\}} |X_n| \, dP < \varepsilon/2$$

and $\delta = \varepsilon/(2\alpha)$ to obtain (1).

'⇐': Put $M = \sup_{n\in\mathbb{N}} \mathrm{E}(|X_n|)$. Then

$$M \geq \int_{\{|X_n|\geq\alpha\}} |X_n|\, dP \geq \alpha \cdot P(\{|X_n| \geq \alpha\}).$$

Hence $P(\{|X_n| \geq \alpha\}) \leq M/\alpha$. Let $\varepsilon > 0$, take $\delta > 0$ according to (1) to obtain for $\alpha > M/\delta$

$$\sup_{n\in\mathbb{N}} \int_{\{|X_n|\geq\alpha\}} |X_n|\, dP < \varepsilon.$$

□

Theorem 1. Let $1 \leq p < \infty$, and assume $X_n \in \mathfrak{L}^p$ for every $n \in \mathbb{N}$. Then

$$(X_n)_{n\in\mathbb{N}} \text{ converges in } \mathfrak{L}^p$$

iff

$$(X_n)_{n\in\mathbb{N}} \text{ converges in probability } \wedge \ (|X_n|^p)_{n\in\mathbb{N}} \text{ is u.i.}$$

Proof. '⇒': Assume $X_n \xrightarrow{\mathfrak{L}^p} X$. From Remark 2.2 we get $X_n \xrightarrow{P} X$. For every $A \in \mathfrak{A}$

$$\|1_A \cdot X_n\|_p \leq \|1_A \cdot (X_n - X)\|_p + \|1_A \cdot X\|_p.$$

Take $A = \Omega$ to obtain $\sup_{n\in\mathbb{N}} \mathrm{E}(|X_n|^p) < \infty$. Let $\varepsilon > 0$, take $k \in \mathbb{N}$ such that

$$\sup_{n>k} \|X_n - X\|_p < \varepsilon. \tag{2}$$

Put $X_0 = 0$. Note that

$$\sup_{0\leq n\leq k} |X_n - X|^p \leq \sum_{n=0}^{k} |X_n - X|^p \in \mathfrak{L}^1.$$

Hence, by Remark 1.(ii),

$$(|X_1 - X|^p, \dots, |X_k - X|^p, |X|^p, |X|^p, \dots) \text{ u.i.}$$

By Lemma 1

$$P(A) < \delta \quad \Rightarrow \quad \sup_{0\leq n\leq k} \|1_A \cdot (X_n - X)\|_p < \varepsilon.$$

for a suitable $\delta > 0$. Together with (2) this implies

$$P(A) < \delta \quad \Rightarrow \quad \sup_{n\in\mathbb{N}} \|1_A \cdot X_n\|_p < 2 \cdot \varepsilon.$$

'⇐': Let $\varepsilon > 0$, put $A = A_{m,n} = \{|X_m - X_n| > \varepsilon\}$. Then

$$\begin{aligned}\|X_m - X_n\|_p &\leq \|1_A \cdot (X_m - X_n)\|_p + \|1_{A^c} \cdot (X_m - X_n)\|_p \\ &\leq \|1_A \cdot X_m\|_p + \|1_A \cdot X_n\|_p + \varepsilon.\end{aligned}$$

By assumption $X_n \stackrel{P}{\longrightarrow} X$ for some $X \in \mathfrak{Z}(\Omega, \mathfrak{A})$. Take $\delta > 0$ according to (1) for $(|X_n|^p)_{n\in\mathbb{N}}$, and note that

$$A_{m,n} \subset \{|X_m - X| > \varepsilon/2\} \cup \{|X_n - X| > \varepsilon/2\}.$$

Hence, for m, n sufficiently large,

$$P(A_{m,n}) < \delta,$$

which implies

$$\|X_m - X_n\|_p \leq 2 \cdot \varepsilon^{1/p} + \varepsilon.$$

Apply Theorem II.6.3. □

Remark 2.

(i) Theorem 1 yields a generalization of Lebesgue's convergence theorem: If $X_n \in \mathfrak{L}^1$ for every $n \in \mathbb{N}$ and $X_n \stackrel{P\text{-a.s.}}{\longrightarrow} X$, then

$$(X_n)_{n\in\mathbb{N}} \text{ u.i.} \quad \Leftrightarrow \quad X \in \mathfrak{L}^1 \wedge X_n \stackrel{\mathfrak{L}^1}{\longrightarrow} X.$$

(ii) Uniform integrability is a property of the distributions only.

Theorem 2.

$$X_n \stackrel{d}{\longrightarrow} X \quad \Rightarrow \quad \mathrm{E}(|X|) \leq \liminf_{n\to\infty} \mathrm{E}(|X_n|).$$

Proof. From Skorohod's Theorem 3.4 we get a probability space $(\widetilde{\Omega}, \widetilde{\mathfrak{A}}, \widetilde{P})$ with random variables $\widetilde{X}_n$, $\widetilde{X}$ such that

$$\widetilde{X}_n \stackrel{\widetilde{P}\text{-a.s.}}{\longrightarrow} \widetilde{X} \quad \wedge \quad \widetilde{P}_{\widetilde{X}_n} = P_{X_n} \quad \wedge \quad \widetilde{P}_{\widetilde{X}} = P_X.$$

Thus $\mathrm{E}(|X|) = \mathrm{E}(|\widetilde{X}|)$ and $\mathrm{E}(|X_n|) = \mathrm{E}(|\widetilde{X}_n|)$. Apply Fatou's Lemma II.5.2. □

Theorem 3. If

$$X_n \stackrel{\mathrm{d}}{\longrightarrow} X \quad \wedge \quad (X_n)_{n\in\mathbb{N}} \text{ u.i.}$$

then

$$X \in \mathfrak{L}^1 \quad \wedge \quad \lim_{n\to\infty} \mathrm{E}(X_n) = \mathrm{E}(X).$$

Proof. Notation as previously. Now $(|\widetilde{X}_n|)_{n\in\mathbb{N}}$ is u.i., see Remark 2.(ii). Hence, by Remark 2.(i), $\widetilde{X} \in \mathfrak{L}^1$ and $\widetilde{X}_n \stackrel{\mathfrak{L}^1}{\longrightarrow} \widetilde{X}$. Thus $\mathrm{E}(|X|) < \infty$ and

$$\lim_{n\to\infty} \mathrm{E}(X_n) = \lim_{n\to\infty} \mathrm{E}(\widetilde{X}_n) = \mathrm{E}(\widetilde{X}) = \mathrm{E}(X).$$

□

Example 2. Example 1 continued. With $X = 0$ we have $X_n \stackrel{P\text{-a.s.}}{\longrightarrow} X$, and therefore $X_n \stackrel{\mathrm{d}}{\longrightarrow} X$. But $\mathrm{E}(X_n) = 1 > 0 = \mathrm{E}(X)$.