2 Convergence in Probability

Motivated by the Examples II.5.2 and II.6.1 we introduce a notion of convergence that is weaker than convergence in mean and convergence almost surely.
In the sequel, X, X_n, etc. random variables on a common probability space $(\Omega, \mathfrak{A}, P)$.

Lemma 1.

$$X_n \overset{P\text{-a.s.}}{\longrightarrow} X \quad \Leftrightarrow \quad \forall\, \varepsilon > 0 : \lim_{n\to\infty} P\left(\left\{\sup_{m\geq n} |X_m - X| > \varepsilon\right\}\right) = 0.$$

Proof. Clearly,

$$\underbrace{\{X_n \to X\}}_{=:A} = \bigcap_{k\in\mathbb{N}} \underbrace{\bigcup_{n\in\mathbb{N}} \underbrace{\bigcap_{m\geq n} \{|X_m - X| \leq 1/k\}}_{=:C_{n,k}}}_{=:B_k} .$$

Hence, $B_k \downarrow A$ and $C_{k,n} \uparrow B_k$. Thus, using the σ-continuity of P,

$$\begin{aligned} & X_n \overset{P\text{-a.s.}}{\longrightarrow} X \\ \Leftrightarrow\ & \forall\, k \in \mathbb{N} :\ P(B_k) = 1 \\ \Leftrightarrow\ & \forall\, k \in \mathbb{N} :\ \lim_{n\to\infty} P(C_{k,n}) = 1 \\ \Leftrightarrow\ & \forall\, k \in \mathbb{N} :\ \lim_{n\to\infty} P\left(\left\{\sup_{m\geq n} |X_m - X| > 1/k\right\}\right) = 0. \end{aligned}$$

□

Definition 1. $(X_n)_n$ *converges* to X *in probability* if

$$\forall\, \varepsilon > 0 :\ \lim_{n\to\infty} P(\{|X_n - X| > \varepsilon\}) = 0.$$

Notation: $X_n \overset{P}{\longrightarrow} X$.

Remark 1. By Lemma 1,

$$X_n \overset{P\text{-a.s.}}{\longrightarrow} X \quad \Rightarrow \quad X_n \overset{P}{\longrightarrow} X.$$

Example II.6.1 shows that '$\Leftarrow$' does not hold in general. The Law of Large Numbers deals with convergence almost surely or convergence in probability, see the introductory Example I.1 and Sections IV.2 and IV.3.

Theorem 1 (Chebyshev-Markov Inequality). For every $\varepsilon > 0$ and every $p \in [1, \infty[$ we have

$$P(|X| > \varepsilon) \leq \frac{1}{\varepsilon^p} \cdot E|X|^p.$$

Proof.

$$E|X|^p = \int_\Omega |X|^p dP \geq \int_{\{|X|\geq\varepsilon\}} |X|^p dP \geq \varepsilon^p \cdot P(\{|X| > \varepsilon\}) .$$

□

Replace X by $X - E(X)$ to derive

Corollary 1 (Chebyshev Inequality, original version). If $\mathrm{E}(X^2) < \infty$, then

$$P(\{|X - \mathrm{E}(X)| \geq \varepsilon\}) \leq \frac{1}{\varepsilon^2} \cdot \mathrm{Var}(X).$$

Theorem 2.

$$d(X, Y) = \int \min(1, |X - Y|)\, dP$$

defines a semi-metric on $\mathfrak{Z}(\Omega, \mathfrak{A})$, and

$$X_n \xrightarrow{P} X \quad \Leftrightarrow \quad \lim_{n \to \infty} d(X_n, X) = 0.$$

Proof. '⇒': Let $X_n \xrightarrow{P} X$. For $\varepsilon > 0$

$$\begin{aligned} &\int \min(1,\, |X_n - X|)\, dP \\ &\quad = \int_{\{|X_n - X| > \varepsilon\}} \min(1,\, |X_n - X|)\, dP + \int_{\{|X_n - X| \leq \varepsilon\}} \min(1,\, |X_n - X|)\, dP \\ &\quad \leq P(\{|X_n - X| > \varepsilon\}) + \min(1, \varepsilon). \end{aligned}$$

'⇐': Let $0 < \varepsilon < 1$. Use Theorem 1 to obtain

$$\begin{aligned} P(\{|X_n - X| > \varepsilon\}) &= P(\{\min(1, |X_n - X|) > \varepsilon\}) \\ &\leq \frac{1}{\varepsilon} \cdot \int \min(1,\, |X_n - X|)\, dP = \frac{1}{\varepsilon} \cdot d(X_n, X). \end{aligned}$$

□

Remark 2. By Theorem 2,

$$X_n \xrightarrow{\mathfrak{L}^p} X \quad \Rightarrow \quad X_n \xrightarrow{P} X.$$

Example II.5.2 shows that '⇐' does not hold in general.

Corollary 2.

$$X_n \xrightarrow{P} X \quad \Rightarrow \quad \exists_{n_k}\, X_{n_k} \xrightarrow{P\text{-a.s.}} X.$$

(Read: There exists a subsequence indexed by n_k, such that..)

Proof. Due to Theorems II.6.3 and 2 there exists a subsequence $(X_{n_k})_{k \in \mathbb{N}}$ such that

$$\min(1, |X_{n_k} - X|) \xrightarrow{P\text{-a.s.}} 0.$$

□

Remark 3. In any semi-metric space (M, d), for any $a_n, a \in M$ we have

$$a_n \to a \quad \Leftrightarrow \quad \forall_{n_k}\, \exists_{n_{k_l}}\, a_{n_{k_l}} \to a\ .$$

This is easily verified by reduction (via $d(a_n, n)$) to convergence of reals to 0, then proof by contradiction.

Corollary 3.

$$X_n \xrightarrow{P} X \quad \Leftrightarrow \quad \forall_{n_k}\, \exists_{n_{k_l}}\; X_{n_{k_\ell}} \xrightarrow{P\text{-a.s.}} X.$$

Proof. '⇒': Corollary 2. '⇐': Remarks 1 and 3 together with Theorem 2. □

Remark 4. We conclude that, in general, there is no semi-metric on $\mathfrak{Z}(\Omega, \mathfrak{A})$ that defines a.s.-convergence. However, if Ω is countable, then

$$X_n \xrightarrow{P\text{-a.s.}} X \quad \Leftrightarrow \quad X_n \xrightarrow{P} X.$$

Proof: `Übung` 8.2.

Lemma 2. Let $\longrightarrow$ denote convergence almost everywhere or convergence in probability. If $X_n^{(i)} \longrightarrow X^{(i)}$ for $i = 1, \dots, k$ and $f : \mathbb{R}^k \to \mathbb{R}$ is continuous, then

$$f \circ (X_n^{(1)}, \dots, X_n^{(k)}) \longrightarrow f \circ (X^{(1)}, \dots, X^{(k)}).$$

Proof. Trivial for convergence almost everywhere, and by Corollary 3 the conclusion holds for convergence in probability, too. □

Corollary 4. Let $X_n \xrightarrow{P} X$. Then

$$X_n \xrightarrow{P} Y \quad \Leftrightarrow \quad X = Y \ P\text{-a.s.}$$

Proof. Corollary 3 and Lemma II.6.1. □