1.2 – Measurable Mappings

Def.1: $(\Omega, \mathfrak{A})$ *measurable space* iff $\Omega \neq \emptyset$ and $\mathfrak{A}$ σ-algebra in Ω. Elements $A \in \mathfrak{A}$ are called ($\mathfrak{A}$–)*measurable sets*.

Let $(\Omega_i, \mathfrak{A}_i)$ measurable spaces, $f : \Omega_1 \to \Omega_2$. For $B \in \mathfrak{A}_2$,

$$\{f \in B\} := f^{-1}(B) := \{\omega \in \Omega_1 \ : \ f(\omega) \in B\} .$$

Rem.1: $f : \Omega_1 \to \Omega_2$

1. $f^{-1}(\mathfrak{A}_2) = \{f^{-1}(A) : A \in \mathfrak{A}_2\}$ is a σ-algebra in Ω_1.
2. $\{A \subset \Omega_2 : f^{-1}(A) \in \mathfrak{A}_1\}$ is a σ-algebra in Ω_2.

Def.2: $f : \Omega_1 \to \Omega_2$ is $\mathfrak{A}_1$-$\mathfrak{A}_2$-*measurable* iff $f^{-1}(\mathfrak{A}_2) \subset \mathfrak{A}_1$, i.e., iff for all $A \in \mathfrak{A}_2$ we have $\{f \in A\} \in \mathfrak{A}_1$.

Thm.1: If

$$(\Omega_1, \mathfrak{A}_1) \xrightarrow{f} (\Omega_2, \mathfrak{A}_2) \xrightarrow{g} (\Omega_3, \mathfrak{A}_3) \ ,$$

f, g measurable, then

$$g \circ f : (\Omega_1, \mathfrak{A}_1) \to (\Omega_3, \mathfrak{A}_3)$$

measurable.

Lemma 1: $f : \Omega_1 \to \Omega_2$, $\mathfrak{E} \subset \mathfrak{P}(\Omega_2)$, then

$$f^{-1}(\sigma(\mathfrak{E})) = \sigma(f^{-1}(\mathfrak{E})).$$

Thm.2: If $f : \Omega_1 \to \Omega_2$ and $\mathfrak{A}_2 = \sigma(\mathfrak{E})$, then

$$f \text{ is } \mathfrak{A}_1\text{-}\mathfrak{A}_2\text{-meas.} \quad \Longleftrightarrow \quad f^{-1}(\mathfrak{E}) \subset \mathfrak{A}_1 \ .$$

Cor.1: Let $(\Omega_i, \mathfrak{G}_i)$ be topological spaces and $f : \Omega_1 \to \Omega_2$ continuous. Then f is $\mathfrak{B}(\Omega_1)$-$\mathfrak{B}(\Omega_2)$-measurable.

Given: $(\Omega_i, \mathfrak{A}_i)_{i\in I}$ meas. spaces, $\Omega \neq \emptyset$, mappings $f_i : \Omega \to \Omega_i$.

Def.3: The *σ-algebra generated by $(f_i)_{i\in I}$ (and $(\mathfrak{A}_i)_{i\in I}$)*

$$\sigma(\{f_i : i \in I\}) = \sigma\Big(\bigcup_{i\in I} f_i^{-1}(\mathfrak{A}_i)\Big).$$

Moreover, set $\sigma(f) = \sigma(\{f\})$.

Rem.2: $\sigma(\{f_i : i \in I\})$ the *smallest* σ-algebra $\mathfrak{A}$ in Ω such that *all mappings* f_i are $\mathfrak{A}$-$\mathfrak{A}_i$-measurable.

Thm.3: Still $f_i : \Omega \to \Omega_i$, and $(\widetilde{\Omega}, \widetilde{\mathfrak{A}})$ meas. space, $g : \widetilde{\Omega} \to \Omega$.

$$\begin{aligned} & g \text{ is } \widetilde{\mathfrak{A}}\text{-}\sigma(\{f_i : i \in I\})\text{-meas.} \\ \Leftrightarrow\ & \forall\, i \in I : f_i \circ g \text{ is } \widetilde{\mathfrak{A}}\text{-}\mathfrak{A}_i\text{-meas.} \end{aligned}$$

$(\Omega, \mathfrak{A})$ meas. space, set

$$\begin{aligned} \mathfrak{Z}(\Omega, \mathfrak{A}) &= \{f : \Omega \to \mathfrak{R} : f \text{ is } \mathfrak{A}\text{-}\mathfrak{B}\text{-measurable}\}, \\ \mathfrak{Z}_+(\Omega, \mathfrak{A}) &= \{f \in \mathfrak{Z}(\Omega, \mathfrak{A}) : f \geq 0\}, \\ \overline{\mathfrak{Z}}(\Omega, \mathfrak{A}) &= \left\{f : \Omega \to \overline{\mathfrak{R}} : f \text{ is } \mathfrak{A}\text{-}\overline{\mathfrak{B}}\text{-measurable}\right\}, \\ \overline{\mathfrak{Z}}_+(\Omega, \mathfrak{A}) &= \left\{f \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A}) : f \geq 0\right\}. \end{aligned}$$

If $f : \Omega \to \mathfrak{R}$, then $f \in \mathfrak{Z}(\Omega, \mathfrak{A})$ iff $f \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$.

Cor.2: For $\prec \in \{\leq, <, \geq, >\}$ and $f : \Omega \to \overline{\mathfrak{R}}$,

$$f \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A}) \quad \Leftrightarrow \quad \forall\, a \in \mathfrak{R} : \{f \prec a\} \in \mathfrak{A}.$$

Thm.4: For $f, g \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$ and $\prec \in \{\leq, <, \geq, >, =, \neq\}$,

$$\{\omega \in \Omega : f(\omega) \prec g(\omega)\} \in \mathfrak{A}.$$

Thm.5: For every sequence $f_1, f_2, \ldots \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$,

1. $\inf_{n\in\mathbb{N}} f_n,\ \sup_{n\in\mathbb{N}} f_n \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$,
2. $\liminf_{n\to\infty} f_n,\ \limsup_{n\to\infty} f_n \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$,
3. if $(f_n)_{n\in\mathbb{N}}$ converges at every point $\omega \in \Omega$, then $\lim_{n\to\infty} f_n \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$.

Set $f^+ = \max(0, f), f^- = \max(0, -f)$.

Rem.3: For $f \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$, $f^+, f^-, |f| \in \overline{\mathfrak{Z}}_+(\Omega, \mathfrak{A})$.

Thm.6: For $f, g \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$,

$$f \pm g,\ f \cdot g,\ f/g \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A}),$$

provided that these functions are well defined.

Def.4: $f \in \mathfrak{Z}(\Omega, \mathfrak{A})$ *simple function* iff $|f(\Omega)| < \infty$. Put

$$\Sigma(\Omega, \mathfrak{A}) = \{f \in \mathfrak{Z}(\Omega, \mathfrak{A}) : f \text{ simple}\},$$
$$\Sigma_+(\Omega, \mathfrak{A}) = \{f \in \Sigma(\Omega, \mathfrak{A}) : f \geq 0\}.$$

Rem.4: $f \in \Sigma(\Omega, \mathfrak{A})$ iff $\exists \alpha_i \in \mathfrak{R}, A_i \in \mathfrak{A}$ s.t.

$$f = \sum_{i=1}^{n} \alpha_i \cdot 1_{A_i}.$$

Thm.7: $f \in \overline{\mathfrak{Z}}_+(\Omega, \mathfrak{A}) \Rightarrow \exists f_1, f_2, \dots \in \Sigma_+(\Omega, \mathfrak{A})$ such that

$$f_n \uparrow f$$

If f bounded, even uniform convergence.

Now $T : \Omega_1 \to \Omega_2$, $\mathfrak{A}_2$ σ–alg. in Ω_2,
$\sigma(T) = T^{-1}(\mathfrak{A}_2)$.

Question: When is a function $f : \Omega_1 \to \overline{\mathfrak{R}}$ $\sigma(T)$–$\overline{\mathfrak{B}}$–measurable?

Thm.8:[Factorization Lemma]

$$f \in \overline{\mathfrak{Z}}(\Omega_1, \sigma(T))$$
$$\Leftrightarrow \exists\, g \in \overline{\mathfrak{Z}}(\Omega_2, \mathfrak{A}_2) \;\;:\;\; f = g \circ T.$$