9 Image Measures

Given: a measure space $(\Omega, \mathfrak{A}, \mu)$, a measurable space $(\Omega', \mathfrak{A}')$, and an $\mathfrak{A}$-$\mathfrak{A}'$-measurable mapping $f : \Omega \to \Omega'$.

Lemma 1.

$$\begin{aligned} f(\mu) : \mathfrak{A}' &\to \mathbb{R}_+ \cup \{\infty\} \\ A' &\mapsto \mu(f^{-1}(A')) = \mu(\{f \in A'\}) \end{aligned}$$

defines a measure on $\mathfrak{A}'$.

Proof. $f(\mu)$ is well-defined, since $f^{-1}(A') \in \mathfrak{A}$ for any $A' \in \mathfrak{A}'$. Further, for $(A_n)_{n\in\mathfrak{N}}$ disjoint, $f^{-1}(A_n)$ are disjoint, too, which implies the σ–additivity. □

Definition 1. $f(\mu)$ is called the *image measure* of μ under f.

Example 1. Let

$$(\Omega, \mathfrak{A}, \mu) = (\mathbb{R}^k, \mathfrak{B}_k, \lambda_k), \qquad (\Omega', \mathfrak{A}') = (\mathbb{R}^k, \mathfrak{B}_k).$$

(i) Fix $a \in \mathbb{R}^k$.For$f(\omega) = \omega + a$ we get

$$f(\lambda_k)(A') = \lambda_k(A' - a) = \lambda_k(A'),$$

see Analysis III ('or' verify this identity for measurable rectangles and apply Theorem 4.4). Thus

$$f(\lambda_k) = \lambda_k.$$

(ii) Fix $r \in \mathbb{R} \setminus \{0\}$. For $f(\omega) = r \cdot \omega$ we get

$$f(\lambda_k)(A') = \lambda_k(1/r \cdot A') = \frac{1}{|r|^k} \cdot \lambda_k(A'),$$

see Analysis III ('or' verify this identity for measurable rectangles and apply Theorem 4.4). Thus

$$f(\lambda_k) = \frac{1}{|r|^k} \cdot \lambda_k.$$

Theorem 1 (Transformation 'Theorem').

(i) for $g \in \overline{\mathfrak{Z}}_+(\Omega', \mathfrak{A}')$

$$\int_{\Omega'} g \, df(\mu) = \int_{\Omega} g \circ f \, d\mu \tag{1}$$

(ii) for $g \in \overline{\mathfrak{Z}}(\Omega', \mathfrak{A}')$

$$g \text{ is } f(\mu)\text{-integrable} \qquad \Leftrightarrow \qquad g \circ f \text{ is } \mu\text{-integrable},$$

in which case (1) holds.

Proof. Algebraic induction: For indicator functions g, both sides are equal by definition; further, both sides obey linearity and monotone convergence in g. □

Example 2. Consider open sets $U, V \subset \mathbb{R}^k$ and a $\mathfrak{C}^1$-diffeomorphism $f : U \to V$. Let

$$(\Omega, \mathfrak{A}, \mu) = (U, U \cap \mathfrak{B}_k, \lambda_k|_{U \cap \mathfrak{B}_k}), \qquad (\Omega', \mathfrak{A}') = (V, V \cap \mathfrak{B}_k).$$

Put

$$\nu = \lambda_k|_{V \cap \mathfrak{B}_k}.$$

Then

$$f(\mu)(A') = \int_{f^{-1}(A')} d\mu = \int_{A'} |\det Df^{-1}| \, d\nu,$$

see Analysis III for the case of an open set $A' \subset V$. Thus

$$f(\mu) = |\det Df^{-1}| \cdot \nu,$$

and therefore

$$\int_U g \circ f \, d\mu = \int_V g \, df(\mu) = \int_V g \cdot |\det Df^{-1}| \, d\nu.$$

Put $g = h \circ f^{-1}$ and $\varphi = f^{-1}$ to obtain

$$\int_U h \, d\mu = \int_V h \circ \varphi \cdot |\det D\varphi| \, d\nu.$$