8 Kernels and Product Measures

Given: measurable spaces $(\Omega_1, \mathfrak{A}_1)$ and $(\Omega_2, \mathfrak{A}_2)$.
Motivation: two-stage experiment. Output $\omega_1 \in \Omega_1$ of the first stage determines probabilistic model for the second stage. Natural idea: Describe ,,conditional probabilities", try to build a model up from this.

Definition 1. $K : \Omega_1 \times \mathfrak{A}_2 \to \overline{\mathbb{R}}$ is a *(Markov) kernel (from* $(\Omega_1, \mathfrak{A}_1)$ *to* $(\Omega_2, \mathfrak{A}_2)$*)*, iff

(i) $K(\omega_1, \cdot)$ is a (probability) measure on $\mathfrak{A}_2$ for every $\omega_1 \in \Omega_1$,

(ii) $K(\cdot, A_2)$ is $\mathfrak{A}_1$-$\overline{\mathfrak{B}}$-measurable for every $A_2 \in \mathfrak{A}_2$.

K is called *σ-finite* kernel iff, additionally, there are $B_i \in \mathfrak{A}_2$ disjoint with

$$\bigcup_{i=1}^{\infty} B_i = \Omega_2 \quad \wedge \quad \forall\, i \in \mathbb{N} : \sup_{\omega_1 \in \Omega_1} K(\omega_1, A_{2,i}) \ < \infty.$$

Example 1. (i) Choose one out of n (unbalanced) coins and throw it once. Parameters $a_1, \ldots, a_n \geq 0$ such that $\sum_{i=1}^{n} a_i = 1$ and $b_1, \ldots, b_n \in [0, 1]$.

Let

$$\Omega_1 = \{1, \ldots, n\}, \qquad \mathfrak{A}_1 = \mathfrak{P}(\Omega_1)$$

and define

$$\mu(\{i\}) = a_i, \qquad i \in \Omega_1,$$

to be the probability of choosing the i-th coin. Moreover, let

$$\Omega_2 = \{\mathrm{H}, \mathrm{T}\}, \qquad \mathfrak{A}_2 = \mathfrak{P}(\Omega_2)$$

and define

$$K(i, \{\mathrm{H}\}) = b_i, \qquad K(i, \{\mathrm{T}\}) = 1 - b_i$$

so $K(i, \{\mathrm{H}\})$ is probability obtaining H when throwing the i-th coin. Formal description:

$$K(i, A_2) = b_i \cdot \delta_{\mathrm{H}}(A_2) + (1 - b_i) \cdot \delta_{\mathrm{T}}(A_2), \qquad A_2 \in \mathfrak{A}_2\ .$$

(ii) (Extremeal case 1) Model for the second stage not influenced by output of the first stage, i.e., for a (probability) measure ν on $\mathfrak{A}_2$

$$\forall\, \omega_1 \in \Omega_1 : \quad K(\omega_1, \cdot) = \nu.$$

In Example 1i this holds if $b_1 = \cdots = b_n$.

(iii) (Extremal case 2) Output of the first stage determines the output of the second stage, i.e., for a $\mathfrak{A}_1$-$\mathfrak{A}_2$-measurable mapping $f : \Omega_1 \to \Omega_2$

$$\forall\, \omega_1 \in \Omega_1 : \quad K(\omega_1, \cdot) = \delta_{f(\omega_1)}.$$

In Example 1i this holds if $b_1, \ldots, b_n \in \{0, 1\}$.

Have: Model for „conditional probabilities" (kernel K), Model for the initial random experiment (probability measure μ on Ω_1). How to build a model for the compound experiment (i.e., probability measure on Ω_2)?
Reasonable, and assumed in the sequel,

$$\Omega = \Omega_1 \times \Omega_2, \qquad \mathfrak{A} = \mathfrak{A}_1 \otimes \mathfrak{A}_2.$$

Question: How to define P?

Example 2. In Example 1i, a reasonable requirement for P is

$$P(\{i\} \times \Omega_2) = a_i = K(a_i, \Omega_2), \qquad P(\{i\} \times \{\mathrm{H}\}) = a_i \cdot b_i = K(i, \{A\})a_i$$

for every $i \in \Omega_1$. Consequently, for $A_2 \subset \Omega_2$

$$P(\{i\} \times A_2) = K(i, A_2) \cdot a_i$$

and for $A \subset \Omega$

$$\begin{aligned} P(A) &= \sum_{i=1}^{n} P(\{(\omega_1, \omega_2) \in A : \omega_1 = i\}) = \sum_{i=1}^{n} P(\{i\} \times \{\omega_2 \in \Omega_2 : (i, \omega_2) \in A\}) \\ &= \sum_{i=1}^{n} K(i, \{(i, \omega_2) \in A\}) \cdot a_i = \int_{\Omega_1} K(i, \{(i, \omega_2) \in A\})\, \mu(di). \end{aligned}$$

May we generally use the right-hand side integral for the definition of P?

Lemma 1. Let $f \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$. Then, for $\omega_1 \in \Omega_1$, the ω_1-*section*

$$f(\omega_1, \cdot) : \Omega_2 \to \overline{\mathbb{R}}$$

of f is $\mathfrak{A}_2$-$\overline{\mathfrak{B}}$-measurable, and for $\omega_2 \in \Omega_2$ the ω_2-*section*

$$f(\cdot, \omega_2) : \Omega_1 \to \overline{\mathbb{R}}$$

of f is $\mathfrak{A}_1$-$\overline{\mathfrak{B}}$-measurable.

Proof. In the case of an ω_1-section. Fix $\omega_1 \in \Omega_1$. Then $\iota_{\omega_1}\Omega_2 \to \Omega_1 \times \Omega_2 : \ \omega_2 \mapsto (\omega_1, \omega_2)$ is $\mathfrak{A}_2$-$\mathfrak{A}$-measurable due to Corollary 3.1.(i); by Theorem 2.1, $f(\omega_1, \cdot) = f \circ \iota_{\omega_1}$ is as well. □

Remark 1. In particular, for $A \in \mathfrak{A}$ and $f = 1_A$

$$f(\omega_1, \cdot) = 1_A(\omega_1, \cdot) = 1_{A_{\omega_1}}$$

where

$$A_{\omega_1} = \{\omega_2 \in \Omega_2 : (\omega_1, \omega_2) \in A\}$$

poor notation is the ω_1-*section* of A. By Lemma 1

$$\forall\, \omega_1 \in \Omega_1 : \quad A_{\omega_1} \in \mathfrak{A}_2.$$

Analogously for the ω_2-*section*

$$A_{\omega_2} = \{\omega_1 \in \Omega_1 : (\omega_1, \omega_2) \in A\}$$

of A.

Given:

- a σ-finite kernel K from $(\Omega_1, \mathfrak{A}_1)$ to $(\Omega_2, \mathfrak{A}_2)$,
- a σ-finite measure μ on $\mathfrak{A}_1$.

Lemma 2. Let $f \in \overline{\mathfrak{Z}}_+$.Then

$$\begin{aligned} g : \Omega_1 &\to \mathbb{R}_+ \cup \{\infty\} \\ \omega_1 &\mapsto \int_{\Omega_2} f(\omega_1, \omega_2)\, K(\omega_1, d\omega_2) \end{aligned}$$

is $\mathfrak{A}_1$-$\mathfrak{B}([0,\infty])$-measurable.

Proof. **Case 1:**

$$\forall\, \omega_1 \in \Omega_1 : K(\omega_1, \Omega_2) < \infty \tag{1}$$

Put $\mathfrak{F} = \{f \in \overline{\mathfrak{Z}}_+ : \text{statement holds for } f\}$

Step1:

$$\forall\, A_1 \in \mathfrak{A}_1,\ A_2 \in \mathfrak{A}_2\ : 1_{A_1 \times A_2} \in \mathfrak{F}$$

Indeed,

$$\int_{\Omega_2} \underbrace{1_{A_1 \times A_2}(\omega_1, \omega_2)}_{=1_{A_1}(\omega_1)\, 1_{A_2}(\omega_2)} K(\omega_1, d\omega_2) = \underbrace{1_{A_1}(\omega_1)}_{\mathfrak{A}_1\text{-}\overline{\mathfrak{B}}\text{-mb}} K(\omega_1, A_2) \tag{2}$$

Step 2:

$$\forall\, A \in \mathfrak{A} : \quad 1_A \in \mathfrak{F}$$

Proof: Set

$$\begin{aligned} \mathfrak{D} &= \{A \in \mathfrak{A} : 1_A \in \mathfrak{F}\} \\ \mathfrak{E} &= \{A_1 \times A_2\ :\ A_1 \in \mathfrak{A}_1, A_2 \in \mathfrak{A}_2\} \end{aligned}$$

Then $\mathfrak{E} \subset \mathfrak{D}$ by Claim 1, $\mathfrak{E}$ closed w.r.t. intersections and $\sigma(\mathfrak{E}) = \mathfrak{A}$

Easy to verify, using (1): $\mathfrak{D}$ is a Dynkin class

Theorem 1.2.(i) yields: $\mathfrak{A} = \sigma(\mathfrak{E}) = \delta(\mathfrak{E}) \subset \mathfrak{D} \subset \mathfrak{A}$, i.e. $\mathfrak{D} = \mathfrak{A}$

Step 3:

$$f_1, f_2 \in \mathfrak{F}\ \wedge\ \alpha \in \mathbb{R}_+ \quad \Rightarrow \quad \alpha f_1 + f_2\ \in \mathfrak{F}$$

Proof: Apply Lemma 5.2, Theorem 2.6

Step 4:

$$f_n \in \mathfrak{F}\ \wedge\ f_n \uparrow f \quad \Rightarrow \quad f \in \mathfrak{F}$$

Proof: Monotone convergence, Theorem 2.5.(iii).

Step 5: Theorem 2.7 implies $\mathfrak{F} = \overline{\mathfrak{Z}}_+$.

Case 2: General Case.
Choose $B_1,\, B_2,\, \ldots\, \in \mathfrak{A}_2$ pairwise disjoint, such that

$$\bigcup_{i=1}^{\infty} B_i = \Omega_2 \quad \wedge \quad \forall\, i \in \mathbb{N} \,:\, \sup_{\omega_1 \in \Omega_1} K(\omega_1, A_{2,i}) < \infty$$

Define $\quad K_i\,(\omega_1,\, \cdot\,) \;=\; K\,(\omega_1,\, \cdot \cap B_i) \;=\; 1_{B_i} \cdot K\,(\omega_1,\, \cdot\,)\,.$

Then

$$\begin{aligned}\int_{\Omega_2} f(\omega_1,\omega_2)\, K(\omega_1,\omega_2) &\overset{\text{Mon. Conv.}}{=} \sum_{i=1}^{\infty} \int_{\Omega_2} 1_{B_i}(\omega_2)\, f(\omega_1,\omega_2)\, K(\omega_1, d\omega_2) \\ &\overset{\text{Thrm 7.2}}{=} \sum_{i=1}^{\infty} \int_{\Omega_2} f(\omega_1,\omega_2)\, K_i(\omega_1, d\omega_2)\end{aligned}$$

Since $\forall\, \omega_1 \in \Omega_1 \;:\; K_i(\omega_1, \Omega_2) \;<\; \infty\,,$

we have $\int_{\Omega_2} f(\,\cdot\,,\omega_2)\, K_i(\,\cdot\,, d\omega_2)$ is $\mathfrak{A}_1$-$\mathfrak{B}([0,\infty])$-measurable.

Apply Theorem 2.6, 2.5

□

Theorem 1. In the above situation,

$$\exists_1 \text{ measure } \nu \text{ on } \mathfrak{A} \text{ such that } \forall\, A_1 \in \mathfrak{A}_1 \;\; \forall\, A_2 \in \mathfrak{A}_2 : \\ \nu(A_1 \times A_2) = \textstyle\int_{A_1} K(\omega_1, A_2)\, \mu(d\omega_1). \tag{3}$$

Moreover, ν is σ-finite, and

$$\forall\, A \in \mathfrak{A} : \quad \nu(A) = \int_{\Omega_1} K(\omega_1, A_{\omega_1})\, \mu(d\omega_1). \tag{4}$$

If μ is a probability measure and K is a Markov kernel then ν is a probability measure, too.

Notation: $\nu = \mu \times K$.

Proof. **Uniqueness:** $\mathfrak{A}_0 = \{A_1 \times A_2 : A_i \in \mathfrak{A}_i\}$ is a $\cap$–closed generator of $\mathfrak{A}$; apply Theorem 4.3.

Existence: Let $A \in \mathfrak{A}, \omega_1 \in \Omega_1$. Then

$$\omega_2 \mapsto K(\omega_1, A_{\omega_1}) = \int_{\Omega_2} \underbrace{1_{A_{\omega_1}}(\omega_2)}_{=1_A(\omega_1,\omega_2)}\, K(\omega_1, d\omega_2)$$

is measurable by Lemma 8.2; hence (4) is well-defined. Moreover, ν defined by (2) is additive, and if $A^{(n)} \uparrow A$, $A^{(n)}, A \in \mathfrak{A}$, then $A^{(n)}_{\omega_1} \uparrow A^{\omega_1}$ for every ω_1, thus $K(\omega_1, A_n^{\omega_1}) \uparrow (\omega_1, A_{\omega_1})$, and by monotone convergence, $\nu(A_n) \uparrow \nu(A)$. Thus, ν is σ–continuous from below, hence a measure.

By virtue of (2), ν satisfies (3). By assumption there are $A_1, A_2, \ldots \in \mathfrak{A}_1$ pairwise disjoint, such that

$$\bigcup_{i=1}^{\infty} A_i = \Omega_1 \quad \wedge \quad \forall\, i \in \mathbb{N} \,:\, \mu(A_i) < \infty$$

and $B_1, B_2, \ldots \in \mathfrak{A}_2$ pairwise disjoint, such that

$$\bigcup_{j=1}^{\infty} B_j = \Omega_2 \quad \wedge \quad \forall\, j \in \mathbb{N} \,:\, \sup_{\omega_1 \in \Omega_1} K(\omega_1, B_j) < \infty$$

Thus $A_i \times B_j$, $i, j \in \mathbb{N}$, pairwise disjoint and $\bigcup_{i,j \in \mathbb{N}} A_i \times B_j = \Omega$,

$$\begin{aligned} (\mu \times K)(A_i \times B_j) &= \int_{A_i} K(\omega_1, B_j)\, \mu(d\omega_1) \\ &\leq \sup_{\omega_1 \in \Omega_1} K(\omega_1, A_j)\, \mu(B_i) \;<\; \infty\,, \end{aligned}$$

i.e., $\mu \times K$ ist σ-finite. □

Example 3. In Example 2 we have $P = \mu \times K$.

Remark 2. Particular case of Theorem 1 with

$$\mu = \mu_1, \qquad \forall\, \omega_1 \in \Omega_1 :\; K(\omega_1, \cdot) = \mu_2$$

for σ-finite measures μ_i on $(\Omega_i, \mathfrak{A}_i)$:

$$\begin{aligned} &\underset{1}{\exists}\,\text{measure } (\mu_1 \times \mu_2) \text{ on } \mathfrak{A} \;\; \forall\, A_1 \in \mathfrak{A}_1 \;\; \forall\, A_2 \in \mathfrak{A}_2 : \\ &\qquad (\mu_1 \times \mu_2)(A_1 \times A_2) = \mu_1(A_1) \cdot \mu_2(A_2). \end{aligned} \tag{5}$$

Moreover, $\mu_1 \times \mu_2$ is σ-finite and satisfies

$$\forall\, A \in \mathfrak{A} : \quad (\mu_1 \times \mu_2)(A) = \int_{\Omega_1} \mu_2(A_{\omega_1})\, \mu(d\omega_1). \tag{6}$$

We add that σ-finiteness is used for the definition (6) and the uniqueness in (5). In general, we only have existence of a measure $\mu_1 \times \mu_2$ with (5). See Elstrodt (1996, §V.1).

Definition 2. $\mu = \mu_1 \times \mu_2$ is called the *product measure* corresponding to μ_1 and μ_2, and $(\Omega, \mathfrak{A}, \mu)$ is called the *product measure space* corresponding to $(\Omega_1, \mathfrak{A}_1, \mu_1)$ and $(\Omega_2, \mathfrak{A}_2, \mu_2)$.

Example 4.

(i) In Example 2 with $b = b_1 = \cdots = b_n$ and $\nu = b \cdot \delta_{\mathrm{H}} + (1 - b) \cdot \delta_{\mathrm{T}}$ we have $P = \mu \times \nu$.

(ii) For countable spaces Ω_i and σ-algebras $\mathfrak{A}_i = \mathfrak{P}(\Omega_i)$ we get

$$\mu_1 \times \mu_2(A) = \sum_{\omega_1 \in \Omega_1} \mu_2(A_{\omega_1}) \cdot \mu_1(\{\omega_1\}), \qquad A \subset \Omega.$$

In particular, for uniform distributions μ_i on finite spaces, $\mu_1 \times \mu_2$ is the uniform distribution on Ω. Cf. Example 3.1 in the case $n = 2$.

(iii) The multi-dimensional Lebesgue measure is a product measure. Namely, for $k, \ell \in \mathbb{N}$ and $A_1 \in \mathfrak{J}_k$, $A_2 \in \mathfrak{J}_\ell$ we have

$$\lambda_{k+\ell}(A_1 \times A_2) = \lambda_k(A_1) \cdot \lambda_\ell(A_2) = (\lambda_k \times \lambda_\ell)(A_1 \times A_2),$$

see Example 4.1.(i). Corollary 4.1 yields

$$\lambda_{k+\ell} = \lambda_k \times \lambda_\ell.$$

From (6) we get

$$\lambda_{k+\ell}(A) = \int_{\mathbb{R}^k} \lambda_\ell(A_{\omega_1})\, \lambda_k(d\omega_1), \qquad A \in \mathfrak{B}_{k+\ell},$$

cf. *Cavalieri's Principle.*

Theorem 2 (Fubini's Theorem).

(i) For $f \in \overline{\mathfrak{Z}}_+(\Omega, \mathfrak{A})$

$$\int_\Omega f\, d(\mu \times K) = \int_{\Omega_1} \int_{\Omega_2} f(\omega_1, \omega_2)\, K(\omega_1, d\omega_2)\, \mu(d\omega_1).$$

(ii) For f $(\mu \times K)$-integrable and

$$A_1 = \{\omega_1 \in \Omega_1 : f(\omega_1, \cdot)\ K(\omega_1, \cdot)\text{-integrable}\}$$

we have

(a) $A_1 \in \mathfrak{A}_1$ and $\mu(A_1^c) = 0$,

(b) $\omega_1 \mapsto \mathbf{1}_{A_1} \cdot \int_{\Omega_2} f(\omega_1, \cdot)\, dK(\omega_1, \cdot)$ is integrable w.r.t. μ,

(c)
$$\int_\Omega f\, d(\mu \times K) = \int_{A_1} \int_{\Omega_2} f(\omega_1, \omega_2)\, K(\omega_1, d\omega_2)\, \mu(d\omega_1).$$

Proof. Ad (i): Algebraic induction: For $f = \mathbf{1}_A$, this is true by definition; both sides are linear in f, hence the claim is true for $f \in \Sigma_+$, and if $f \in \overline{\mathfrak{Z}}_+$, there are $f_n \in \Sigma_+$ with $f_n \uparrow f$. Now for each fixed ω_1, $f_n(\omega_1, \cdot) \uparrow f$, hence by monoton convergence,

$$\int_{\Omega_2} f_n(\omega_1, \omega_2) K(\omega_1, d\omega_2) \uparrow \int_{\Omega_2} f(\omega_1, \omega_2) K(\omega_1, d\omega_2)\ ,$$

and again by monotone convergence

$$\int_{\Omega_1}\int_{\Omega_2} f_n(\omega_1,\omega_2)K(\omega_1,d\omega_2) \uparrow \int_{\Omega_1}\int_{\Omega_2} f(\omega_1,\omega_2)K(\omega_1,d\omega_2)\ .$$

Ad (ii): By (i), we have, for $f_\pm = \max 0, \pm f$,

$$\int_\Omega f_\pm\, d(\mu\times K) = \int_{\Omega_1}\int_{\Omega_2} f_\pm(\omega_1,\omega_2)\, K(\omega_1,d\omega_2)\, \mu(d\omega_1).$$

Then

$$A_1^\pm := \left\{\omega_1\ :\ \int_{\Omega_2} f_\pm(\omega_1,\omega_2)\, K(\omega_1,d\omega_2) < \infty\right\}$$

is in $\mathfrak{A}_1$ by Lemma 8.2, and $A_1 = A^+ \cap A^-$. Moreover, $\mu((A^\pm)^c) < \infty$ by Theorem 5.4 and part (i). Part (b) and (c) follow immediately, since they are true for $f_\pm$. □

Remark 3. For brevity, we write

$$\int_{\Omega_1}\int_{\Omega_2} f(\omega_1,\omega_2)\, K(\omega_1,d\omega_2)\, \mu(d\omega_1) = \int_{A_1}\int_{\Omega_2} f(\omega_1,\omega_2)\, K(\omega_1,d\omega_2)\, \mu(d\omega_1),$$

if f is $(\mu\times K)$-integrable. For $f \in \overline{\mathfrak{Z}}(\Omega,\mathfrak{A})$

$$f \text{ is } (\mu\times K)\text{-integrable} \qquad \Leftrightarrow \qquad \int_{\Omega_1}\int_{\Omega_2} |f|(\omega_1,\omega_2)\, K(\omega_1,d\omega_2)\, \mu(d\omega_1) < \infty.$$

Corollary 1 (Fubini's Theorem). For σ-finite measures μ_i on $\mathfrak{A}_i$ and a $(\mu_1\times\mu_2)$-integrable function f

$$\begin{aligned}\int_\Omega f\, d(\mu_1\times\mu_2) &= \int_{\Omega_1}\int_{\Omega_2} f(\omega_1,\omega_2)\, \mu_2(d\omega_2)\, \mu_1(d\omega_1)\\ &= \int_{\Omega_2}\int_{\Omega_1} f(\omega_1,\omega_2)\, \mu_1(d\omega_1)\, \mu_2(d\omega_2).\end{aligned}$$

Proof. Theorem 2 yields the first equality. For the second equality, put $\widetilde{f}(\omega_2,\omega_1) = f(\omega_1,\omega_2)$ and note that $\int_\Omega f\, d(\mu_1\times\mu_2) = \int_\Omega \widetilde{f}\, d(\mu_2\times\mu_1)$. □

Corollary 2. For every measurable space $(\Omega,\mathfrak{A})$, every σ-finite measure μ on $\mathfrak{A}$, and every $f \in \overline{\mathfrak{Z}}_+(\Omega,\mathfrak{A})$

$$\int_\Omega f\, d\mu = \int_{]0,\infty[} \mu(\{f > x\})\, \lambda_1(dx).$$

Proof. Übung 6.2. □

Now we construct a stochastic model for a series of experiments, where the outputs of the first $i-1$ stages determine the model for the ith stage. We simply iterate our two–step procedure.

Given: measurable spaces $(\Omega_i,\mathfrak{A}_i)$ for $i \in I$, where $I = \{1,\dots,n\}$ or $I = \mathbb{N}$. Put

$$(\Omega_i',\mathfrak{A}_i') = \left(\prod_{j=1}^{i} \Omega_j, \bigotimes_{j=1}^{i} \mathfrak{A}_j\right),$$

and note that

$$\prod_{j=1}^{i} \Omega_j = \Omega'_{i-1} \times \Omega_i \qquad \wedge \qquad \bigotimes_{j=1}^{i} \mathfrak{A}_j = \mathfrak{A}'_{i-1} \otimes \mathfrak{A}_i$$

for $i \in I \setminus \{1\}$. Furthermore, let

$$\Omega = \prod_{i \in I} \Omega_i, \qquad \mathfrak{A} = \bigotimes_{i \in I} \mathfrak{A}_i. \tag{7}$$

Given:

- σ-finite kernels K_i from $\left(\Omega'_{i-1}, \mathfrak{A}'_{i-1}\right)$ to $(\Omega_i, \mathfrak{A}_i)$ for $i \in I \setminus \{1\}$,
- a σ-finite measure μ on $\mathfrak{A}_1$.

Theorem 3. For $I = \{1, \dots, n\}$

$$\begin{aligned} &\underset{1}{\exists} \text{ measure } \nu \text{ on } \mathfrak{A} \ \ \forall A_1 \in \mathfrak{A}_1 \dots \forall A_n \in \mathfrak{A}_n : \\ &\quad \nu(A_1 \times \cdots \times A_n) \\ &= \int_{A_1} \dots \int_{A_{n-1}} K_n((\omega_1, \dots, \omega_{n-1}), A_n)\, K_{n-1}((\omega_1, \dots, \omega_{n-2}), d\omega_{n-1}) \cdots \mu(d\omega_1). \end{aligned}$$

Moreover, ν is σ-finite and for f ν-integrable (the short version)

$$\int_{\Omega} f \, d\nu = \int_{\Omega_1} \dots \int_{\Omega_n} f(\omega_1, \dots, \omega_n) K_n((\omega_1, \dots, \omega_{n-1}), d\omega_n) \cdots \mu(d\omega_1). \tag{8}$$

Notation: $\nu = \mu \times K_2 \times \cdots \times K_n$.

Proof. Induction on n, using Theorems 1 and 2. □

Remark 4. Particular case of Theorem 3 with

$$\mu = \mu_1, \qquad \forall\, i \in I \setminus \{1\} \ \forall\, \omega'_{i-1} \in \Omega'_{i-1} : \ K_i(\omega'_{i-1}, \cdot) = \mu_i \tag{9}$$

for σ-finite measures μ_i on $\mathfrak{A}_i$:

$$\begin{aligned} &\underset{1}{\exists} \text{ measure } \mu_1 \times \cdots \times \mu_n \text{ on } \mathfrak{A} \ \ \forall A_1 \in \mathfrak{A}_1 \dots \forall A_n \in \mathfrak{A}_n : \\ &\qquad \mu_1 \times \cdots \times \mu_n(A_1 \times \cdots \times A_n) = \mu_1(A_1) \cdot \dots \cdot \mu_n(A_n). \end{aligned}$$

Moreover, $\mu_1 \times \cdots \times \mu_n$ is σ-finite and for every $\mu_1 \times \cdots \times \mu_n$-integrable function f

$$\int_{\Omega} f \, d(\mu_1 \times \cdots \times \mu_n) = \int_{\Omega_1} \dots \int_{\Omega_n} f(\omega_1, \dots, \omega_n)\ \mu_n(d\omega_n) \cdots d\mu_1(\omega_1).$$

Definition 3. $\mu = \mu_1 \times \cdots \times \mu_n$ is called the *product measure* corresponding to μ_i for $i = 1, \dots, n$, and $(\Omega, \mathfrak{A}, \mu)$ is called the *product measure space* corresponding to $(\Omega_i, \mathfrak{A}_i, \mu_i)$ for $i = 1, \dots, n$.

Example 5.

(i) For uniform distributions μ_i on finite spaces Ω_i, $\mu_1 \times \cdots \times \mu_n$ is the uniform distribution on Ω. Cf. Example 3.1 in the case $n \in \mathbb{N}$.

(ii)

$$\lambda_n = \lambda_1 \times \cdots \times \lambda_1.$$

Theorem 4 (Ionescu-Tulcea). Assume that μ is a probability measure and that K_i are Markov kernels for $i \in \mathbb{N} \setminus \{1\}$. Then, for $I = \mathbb{N}$,

$$\underset{1}{\exists} \text{probability measure } P \text{ on } \mathfrak{A} \ \ \forall n \in N \ \ \forall A_1 \in \mathfrak{A}_1 \ldots \forall A_n \in \mathfrak{A}_n :$$

$$P\Big(A_1 \times \cdots \times A_n \times \prod_{i=n+1}^{\infty} \Omega_i\Big) = (\mu \times K_2 \times \cdots \times K_n)(A_1 \times \cdots \times A_n). \tag{10}$$

Proof. **Uniqueness:** By (10), P is uniquely determined on the class of measurable rectangles. Apply Theorem 4.4.

Existence: On the semi–algebra of measurable rectangles we define P by (10). By (8), one easily checks that this is well–defined and, by definition, additive. By Theorem 4.2, P is extended uniquely to a content on the algebra of cylinder sets, still denoted by P. Obviously,

$$P(A \times \prod_{j>n} \Omega_j) = (\mu \times K_2 \cdots \times K_n)(A) , \qquad A \in \bigotimes_{j \le n} \mathfrak{A}_j .$$

We claim that this content is σ–additive; then, by Corollary 4.1, there is a unique extension to $\mathfrak{A}$. By Theorem 4.1, it suffices to show that P is σ–continuous at $\emptyset$. So let A_n be cylinder sets, $A_n \downarrow \emptyset$, and assume $\lim_n P(A_n) > 0$. Without loss of generality, we may assume

$$A_n = \big\{(\omega_i)_{i\in\mathbb{N}} \ : \ (\omega_1, \ldots, \omega_n) \in B_n\big\}$$

for some B_n. Set $\omega^i = (\omega_1, \ldots, \omega_i)$. By (8) and Theorem 2,

$$P(A_n) = \int_{\Omega_1} \underbrace{\int_{\Omega_2} \cdots \int_{\Omega_n} \mathbf{1}_{B_n}(\omega^n) K_n(\omega^{n-1}, d\omega_n) \ldots K_1(\omega_1, d\omega_2)}_{=: f_n^{(1)}(\omega_1)} \, d\mu(\omega_1) = \int_{\Omega_1} f_n^{(1)}(\omega_1) d\mu(\omega_1) .$$

Since $A_{n+1} \subseteq A_n$, $B_{n+1} \subseteq B_n \times \Omega$, and hence

$$\mathbf{1}_{B_{n+1}}(\omega^{n+1}) \le \mathbf{1}_{B_n}(\omega^n) ,$$

thus, the monotonicity of integrals show that $f_n^{(1)}$ is monotonically decreasing; set $f^{(1)} = \lim_n f_n^{(1)}$. By Lebesgue's theorem (1 is a majorant),

$$0 < \lim_n P(A_n) = \int_{\Omega_1} f(\omega_1) d\mu(\omega_1) .$$

In particular, there is $\hat{\omega}_1$ with $f^{(1)}(\hat{\omega}_1) > 0$. In particular, $\omega_1 \in B_1$.

Next, $K_2(\hat{\omega}_1, \cdot)$ is a probability measure on Ω_2, and for $n > 2$ we define

$$f_n^{(2)}(\omega_2) := \int_{\Omega_3} \cdots \int_{\Omega_n} \mathbf{1}_{B_n}(\hat{\omega}_1, \omega_2, \ldots, \omega_n) K_n((\hat{\omega}_1, \ldots, \omega_{n-1}, d\omega_n), \ldots K_3(\hat{\omega}_1, \omega_2, d\omega_3)) .$$

Then

$$f_n^{(1)}(\hat{\omega}_1) = \int_{\Omega_2} f_n^{(2)}(\omega_2) K_2(\hat{\omega}_1, d\omega_2) \; ;$$

again $f_n^{(2)}$ is monotonely decreasing against some $f^{(2)}$, and by Lebesgue

$$0 < f^{(1)}(\hat{\omega}_1) = \int_{\Omega_2} f^{(2)}(\omega_2) K(\hat{\omega}_1, d\omega_2) \; .$$

Thus there is $\hat{\omega_2}$ with $f^{(2)}(\hat{\omega}_2) > 0$, i.e., $(\hat{\omega}_1, \hat{\omega_2}) \in B_2$. Iterating this procedure, one finds a sequence $\hat{\omega}$ with $(\hat{\omega}_1, \dots, \hat{\omega}_n) \in B_n$ for all n, i.e., $\hat{\omega} \in \bigcap A_n = \emptyset$, a contradiction. □

Example 6. The queueing model, see Übung6.3. Here $K_i(\omega_1, \dots, \omega_{i-1}), \cdot)$ only depends on ω_{i-1}. Outlook: Markov processes.

Given: a non-empty *arbitrary* index set I and probability spaces $(\Omega_i, \mathfrak{A}_i, \mu_i)$ for $i \in I$. Recall the definition (7).

Theorem 5.

$$\underset{1}{\exists} \text{probability measure } P \text{ on } \mathfrak{A} \;\; \forall S \in \mathfrak{P}_0(I) \;\; \forall A_i \in \mathfrak{A}_i, \; i \in S :$$

$$P\Big(\prod_{i \in S} A_i \times \prod_{i \in I \setminus S} \Omega_i\Big) = \prod_{i \in S} \mu_i(A_i). \tag{11}$$

Notation: $P = \prod_{i \in I} \mu_i$.

Proof. See Remark 4 in the case of a finite set I.

If $|I| = |\mathbb{N}|$, assume $I = \mathbb{N}$ without loss of generality. The particular case of Theorem 4 with (9) for probability measures μ_i on $\mathfrak{A}_i$ shows

$$\underset{1}{\exists} \text{probability measure } P \text{ on } \mathfrak{A} \;\; \forall n \in \mathbb{N} \;\; \forall A_1 \in \mathfrak{A}_1 \dots \forall A_n \in \mathfrak{A}_n :$$

$$P\Big(A_1 \times \dots \times A_n \times \prod_{i=n+1}^{\infty} \Omega_i\Big) = \mu_1(A_1) \cdot \dots \cdot \mu_n(A_n).$$

If I is uncountable, we use Theorem 3.2. For $S \subset I$ non-empty and countable and for $B \in \bigotimes_{i \in S} \mathfrak{A}_i$ we put

$$P\big(\big(\pi_S^I\big)^{-1} B\big) = \prod_{i \in S} \mu_i(B).$$

Hereby we get a well-defined mapping $P : \mathfrak{A} \to \mathbb{R}$, which clearly is a probability measure and satisfies (11). Use Theorem 4.4 to obtain the uniqueness result. □

Definition 4. $P = \prod_{i \in I} \mu_i$ is called the *product measure* corresponding to μ_i for $i \in I$, and $(\Omega, \mathfrak{A}, P)$ is called the *product measure space* corresponding to $(\Omega_i, \mathfrak{A}_i, \mu_i)$ for $i \in I$.

Remark 5. Theorem 5 answers the question that is posed in Example 3.1 in full generality. Moreover, it is the basis for a positive answer to the question from the introductory Example I.2, see Theorem III.5.2.