6 $\mathfrak{L}^p$-Spaces

Given: a measure space $(\Omega, \mathfrak{A}, \mu)$ and $1 \le p < \infty$. Put $\mathfrak{Z} = \mathfrak{Z}(\Omega, \mathfrak{A})$.

Definition 1.

$$\mathfrak{L}^p = \mathfrak{L}^p(\Omega, \mathfrak{A}, \mu) = \left\{ f \in \mathfrak{Z} : \int |f|^p \, d\mu < \infty \right\}.$$

In particular, for $p = 1$: *integrable functions* and $\mathfrak{L} = \mathfrak{L}^1$, and for $p = 2$: *square-integrable functions*. Put

$$\|f\|_p = \left(\int |f|^p \, d\mu \right)^{1/p}, \qquad f \in \mathfrak{L}^p.$$

Theorem 1 (Hölder inequality). Let $1 < p, q < \infty$ such that $1/p + 1/q = 1$ and let $f \in \mathfrak{L}^p$, $g \in \mathfrak{L}^q$. Then

$$\int |f \cdot g| \, d\mu \le \|f\|_p \cdot \|g\|_q.$$

In particular, for $p = q = 2$: *Cauchy-Schwarz inequality.*

Proof. See Analysis III or Elstrodt (1996, §VI.1) as well as Theorem 5.3. □

Theorem 2. $\mathfrak{L}^p$ is a vector space and $\|\cdot\|_p$ is a semi-norm on $\mathfrak{L}^p$. Furthermore,

$$\|f\|_p = 0 \quad \Leftrightarrow \quad f = 0 \ \mu\text{-a.e.}$$

Proof. See Analysis III or Elstrodt (1996, §VI.2). □

Definition 2. Let $f, f_n \in \mathfrak{L}^p$ for $n \in \mathbb{N}$. $(f_n)_n$ *converges to* f *in* $\mathfrak{L}^p$ *(in mean of order* p*)* if

$$\lim_{n \to \infty} \|f - f_n\|_p = 0.$$

In particular, for $p = 1$: *convergence in mean*, and for $p = 2$: *mean-square convergence*. Notation:

$$f_n \xrightarrow{\mathfrak{L}^p} f.$$

Remark 1. Let $f, f_n \in \overline{\mathfrak{Z}}$ for $n \in \mathbb{N}$. Recall (define) that $(f_n)_n$ *converges to* f μ*-a.e.* if

$$\mu(A^c) = 0$$

for

$$A = \left\{ \lim_{n \to \infty} f_n = f \right\} = \left\{ \limsup_{n \to \infty} f_n = \liminf_{n \to \infty} f_n \right\} \cap \left\{ \limsup_{n \to \infty} f_n = f \right\} \in \mathfrak{A}.$$

Notation:

$$f_n \xrightarrow{\mu\text{-a.e.}} f.$$

Lemma 1. Let $f, g, f_n \in \mathfrak{L}^p$ for $n \in \mathbb{N}$ such that $f_n \xrightarrow{\mathfrak{L}^p} f$. Then

$$f_n \xrightarrow{\mathfrak{L}^p} g \quad \Leftrightarrow \quad f = g \ \mu\text{-a.e.}$$

Analogously for convergence almost everywhere.

Proof. For convergence in $\mathfrak{L}^p$: '$\Leftarrow$' follows from Theorem 5.4.(ii). Use

$$\|f - g\|_p \leq \|f - f_n\|_p + \|f_n - g\|_p$$

to verify '$\Rightarrow$'.
For convergence almost everywhere: '$\Leftarrow$' trivially holds. Use

$$\left\{\lim_{n\to\infty} f_n = f\right\} \cap \left\{\lim_{n\to\infty} f_n = g\right\} \subset \{f = g\}$$

to verify '$\Rightarrow$'. $\square$

Theorem 3 (Fischer-Riesz). Consider a sequence $(f_n)_n$ in $\mathfrak{L}^p$. Then

(i) $(f_n)_n$ Cauchy sequence $\Rightarrow \exists\, f \in \mathfrak{L}^p: \; f_n \xrightarrow{\mathfrak{L}^p} f$ (*completeness*),

(ii) $f_n \xrightarrow{\mathfrak{L}^p} f \Rightarrow \exists\,$subsequence $(f_{n_k})_k : f_{n_k} \xrightarrow{\mu\text{-a.e.}} f$.

Proof. Ad (i): Consider a Cauchy sequence $(f_n)_n$ and a subsequence $(f_{n_k})_k$ such that

$$\forall\, k \in \mathbb{N}\; \forall\, m \geq n_k : \|f_m - f_{n_k}\|_p \leq 2^{-k}.$$

For

$$g_k = f_{n_{k+1}} - f_{n_k} \in \mathfrak{L}^p$$

we have

$$\Big\| \sum_{\ell=1}^{k} |g_\ell| \Big\|_p \leq \sum_{\ell=1}^{k} \|g_\ell\|_p \leq \sum_{\ell=1}^{k} 2^{-\ell} \leq 1.$$

Put $g = \sum_{\ell=1}^{\infty} |g_\ell| \in \overline{\mathfrak{Z}}_+$. By Theorem 5.1

$$\int g^p \, d\mu = \int \sup_k \Big(\sum_{\ell=1}^{k} |g_\ell| \Big)^p d\mu = \sup_k \int \Big(\sum_{\ell=1}^{k} |g_\ell| \Big)^p d\mu \leq 1. \tag{1}$$

Thus, in particular, $\sum_{\ell=1}^{\infty} |g_\ell|$ and $\sum_{\ell=1}^{\infty} g_\ell$ converge μ-a.e., see Theorem 5.4.(i). Since

$$f_{n_{k+1}} = \sum_{\ell=1}^{k} g_\ell + f_{n_1},$$

we have

$$f = \lim_{k \to \infty} f_{n_k} \; \mu\text{-a.e.}$$

for some $f \in \mathfrak{Z}$. Furthermore,

$$|f - f_{n_k}| \leq \sum_{\ell=k}^{\infty} |g_\ell| \leq g \; \mu\text{-a.e.},$$

so that, by Theorem 5.5 and (1),

$$\lim_{k \to \infty} \int |f - f_{n_k}|^p \, d\mu = 0.$$

It follows that

$$\lim_{n \to \infty} \|f - f_n\|_p = 0,$$

too. Finally, by Theorem 2, $f \in \mathfrak{L}^p$.
Ad (ii): Assume that

$$f_n \overset{\mathfrak{L}^p}{\to} f.$$

According to the proof of (i) there exists $\widetilde{f} \in \mathfrak{L}^p$ and a subsequence $(f_{n_k})_k$ such that

$$f_{n_k} \xrightarrow{\mu\text{-a.e.}} \widetilde{f} \wedge f_{n_k} \xrightarrow{\mathfrak{L}^p} \widetilde{f}.$$

Use Lemma 1. □

Example 1. Let $(\Omega, \mathfrak{A}, \mu) = \big([0,1], \mathfrak{B}([0,1]), \lambda_1|_{\mathfrak{B}([0,1])}\big)$. (By Remark 1.7.(ii) we have $\mathfrak{B}([0,1]) \subset \mathfrak{B}_1$). Define

$$A_1 = [0,1]$$
$$A_2 = [0,1/2], \quad A_3 = [1/2,1]$$
$$A_4 = [0,1/3], \quad A_5 = [1/3,2/3], \quad A_6 = [2/3,1]$$
$$\text{etc.}$$

Put $f_n = 1_{A_n}$. Then

$$\lim_{n\to\infty} \|f_n - 0\|_p = \lim_{n\to\infty} \|f_n\|_p = 0 \tag{2}$$

but

$$\{(f_n)_n \text{ converges}\} = \emptyset.$$

Remark 2. Define

$$\mathfrak{L}^\infty = \mathfrak{L}^\infty(\Omega, \mathfrak{A}, P) = \{f \in \mathfrak{Z} : \exists\, c \in \mathbb{R}_+ : |f| \le c \ \mu\text{-a.e.}\}$$

and

$$\|f\|_\infty = \inf\{c \in \mathbb{R}_+ : |f| \le c \ \mu\text{-a.e.}\}, \qquad f \in \mathfrak{L}^\infty.$$

$f \in \mathfrak{L}^\infty$ is called *essentially bounded* and $\|f\|_\infty$ is called the *essential supremum of* $|f|$. Use Theorem 4.1.(iii) to verify that

$$|f| \le \|f\|_\infty \ \mu\text{-a.e.}$$

The definitions and results of this section, except (2), extend to the case $p = \infty$, where $q = 1$ in Theorem 1. In Theorem 3.(ii) we even have $f_n \xrightarrow{\mathfrak{L}^\infty} f \Rightarrow f_n \xrightarrow{\mu\text{-a.e.}} f$.

Remark 3. Put

$$\mathfrak{N}^p = \{f \in \mathfrak{L}^p : f = 0 \ \mu\text{-a.e.}\}$$

Then the quotient space $L^p = \mathfrak{L}^p/\mathfrak{N}^p$ is a Banach space. In particular, for $p = 2$, L^2 is a Hilbert space, with semi-inner product on $\mathfrak{L}^2$ given by

$$\langle f, g\rangle = \int f \cdot g \, d\mu, \qquad f, g \in \mathfrak{L}^2.$$

Theorem 4. If μ is finite and $1 \le p < q \le \infty$ then

$$\mathfrak{L}^q \subset \mathfrak{L}^p$$

and

$$\|f\|_p \le \mu(\Omega)^{1/p-1/q} \cdot \|f\|_q, \qquad f \in \mathfrak{L}^q.$$

Proof. The result trivially holds for $q = \infty$.In the sequel, $q < \infty$. Use $|f|^p \le 1 + |f|^q$ and Theorem 5.4.(iii) to obtain $\mathfrak{L}^q \subset \mathfrak{L}^p$. Put $r = q/p$ and define s by $1/r + 1/s = 1$. Theorem 1 yields

$$\int |f|^p \, d\mu \le \left(\int |f|^{p\cdot r} \, d\mu\right)^{1/r} \cdot \big(\mu(\Omega)\big)^{1/s}.$$

□

Example 2. Let$1 \le p < q \le \infty$. With respect to the counting measure on $\mathfrak{P}(\mathbb{N})$, $\mathfrak{L}^p \subset \mathfrak{L}^q$. With respect to the Lebesgue measure on $\mathfrak{B}_k$ neither $\mathfrak{L}^q \subset \mathfrak{L}^p$ nor $\mathfrak{L}^p \subset \mathfrak{L}^q$.