5 Integration

For the proofs, see Analysis IV or Elstrodt (1996, Kap. VI).
Fixed in this section: A measure space $(\Omega, \mathfrak{A}, \mu)$. Notation:

- $\Sigma_+ = \Sigma_+(\Omega, \mathfrak{A})$ (nonnegative simple functions),
- $\overline{\mathfrak{Z}}_+ = \overline{\mathfrak{Z}}_+(\Omega, \mathfrak{A})$ (nonnegative $\mathfrak{A}$-$\overline{\mathfrak{B}}$-measurable functions),

Definition 1. *Integral* Let $f \in \Sigma_+$,

$$f = \sum_{i=1}^{n} \alpha_i \cdot 1_{A_i} \ , \qquad \alpha_i \in \mathfrak{R}, A_i \in \mathfrak{A} \ .$$

Then define its *Integral* w.r.t. μ as

$$\int f \, d\mu = \sum_{i=1}^{n} \alpha_i \cdot \mu(A_i) \ .$$

Lemma 1. The mapping $\int \cdot \mathrm{d}\mu : \Sigma_+ \to \mathfrak{R}_+$ is

(i) positive–linear: $\int(\alpha f + \beta g) \, d\mu = \alpha \int f \, d\mu + \beta \int g \, d\mu$, $f, g \in \Sigma_+$, $\alpha, \beta \in \mathfrak{R}_+$,

(ii) monotone: $f \leq g \Rightarrow \int f \, d\mu \leq \int g \, d\mu$ (*monotonicity*).

Definition 2. *Integral* of $f \in \overline{\mathfrak{Z}}_+$ w.r.t. μ

$$\int f \, d\mu = \sup \Big\{ \int g \, d\mu : g \in \Sigma_+ \wedge g \leq f \Big\}.$$

Theorem 1 (Monotone convergence, Beppo Levi). (e.g., Thm.6.4, Analysis IV, SS06) Let $f_n \in \overline{\mathfrak{Z}}_+$ such that

$$\forall \, n \in \mathbb{N} : f_n \leq f_{n+1}.$$

Then

$$\int \sup_n f_n \, d\mu = \sup_n \int f_n \, d\mu.$$

Remark 1. For every $f \in \overline{\mathfrak{Z}}_+$ there exists a sequence of functions $f_n \in \Sigma_+$ such that $f_n \uparrow f$, see Theorem 2.7.

Example 1. Consider

$$f_n = \frac{1}{n} \cdot 1_{[0,n]}$$

on $(\mathbb{R}, \mathfrak{B}, \lambda_1)$. Then

$$\int f_n \, d\lambda_1 = 1, \qquad \lim_{n \to \infty} f_n = 0.$$

Lemma 2. The mapping $\int \cdot \, \mathrm{d}\mu \ : \ \mathfrak{Z}_+ \to \overline{\mathfrak{R}}_+$ is still positive–linear and monotone.

Theorem 2 (Fatou's Lemma). (See, e.g., Lemma 6.6, Ananlysis IV, SS06) For every sequence $(f_n)_n$ in $\overline{\mathfrak{Z}}_+$

$$\int \liminf_{n\to\infty} f_n \, d\mu \le \liminf_{n\to\infty} \int f_n \, d\mu.$$

Proof. For $g_n = \inf_{k\ge n} f_k$ we have $g_n \in \overline{\mathfrak{Z}}_+$ and $g_n \uparrow \liminf_n f_n$. By Theorem 1 and Lemma 1.(iii)

$$\int \liminf_n f_n \, d\mu = \lim_{n\to\infty} \int g_n \, d\mu \le \liminf_{n\to\infty} \int f_n \, d\mu.$$

□

Theorem 3. Let $f \in \overline{\mathfrak{Z}}_+$. Then

$$\int f \, d\mu = 0 \Leftrightarrow \mu(\{f > 0\}) = 0.$$

Definition 3. A property Π holds *μ-almost everywhere (μ-a.e., a.e.)*, if

$$\exists\, A \in \mathfrak{A} : \{\omega \in \Omega : \Pi \text{ does not hold for } \omega\} \subset A \wedge \mu(A) = 0.$$

In case of a probability measure we say: *μ-almost surely, μ-a.s., with probability one*.

Notation: $\overline{\mathfrak{Z}} = \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$ is the class of $\mathfrak{A}$-$\overline{\mathfrak{B}}$-measurable functions.

Definition 4. $f \in \overline{\mathfrak{Z}}$ *quasi-μ-integrable* if

$$\int f_+ \, d\mu < \infty \quad \vee \quad \int f_- \, d\mu < \infty.$$

In this case: *integral* of f (w.r.t. μ)

$$\int f \, d\mu = \int f_+ \, d\mu - \int f_- \, d\mu.$$

$f \in \overline{\mathfrak{Z}}$ *μ-integrable* if

$$\int f_+ \, d\mu < \infty \quad \wedge \quad \int f_- \, d\mu < \infty.$$

Theorem 4.

(i) f μ-integrable $\Rightarrow \mu(\{|f| = \infty\}) = 0$,

(ii) f μ-integrable $\wedge\; g \in \overline{\mathfrak{Z}} \wedge f = g$ μ-a.e. $\Rightarrow g$ μ-integrable $\wedge \int f \, d\mu = \int g \, d\mu$.

(iii) equivalent properties for $f \in \overline{\mathfrak{Z}}$:

  (a) f μ-integrable,
  (b) $|f|$ μ-integrable,
  (c) $\exists\, g : g$ μ-integrable $\wedge\; |f| \le g$ μ-a.e.,

(iv) for f and g μ-integrable and $c \in \mathbb{R}$

   (a) $f+g$ well-defined μ-a.e. and μ-integrable with $\int(f+g)\,d\mu = \int f\,d\mu + \int g\,d\mu$,

   (b) $c \cdot f$ μ-integrable with $\int(cf)\,d\mu = c \cdot \int f\,d\mu$,

   (c) $f \leq g$ μ-a.e. $\Rightarrow \int f\,d\mu \leq \int g\,d\mu$.

Theorem 5 (Dominated convergence, Lebesgue). Assume that

(i) $f_n \in \overline{\mathfrak{Z}}$ for $n \in \mathbb{N}$,

(ii) $\exists\, g$ μ-integrable $\forall\, n \in \mathbb{N} : |f_n| \leq g$ μ-a.e.,

(iii) $f \in \overline{\mathfrak{Z}}$ such that $\lim_{n\to\infty} f_n = f$ μ-a.e.

Then f is μ-integrable and

$$\int f\,d\mu = \lim_{n\to\infty} \int f_n\,d\mu.$$

Example 2. Consider

$$f_n = n \cdot 1_{]0,1/n[}$$

on $(\mathbb{R}, \mathfrak{B}, \lambda_1)$. Then

$$\int f_n\,d\lambda_1 = 1, \qquad \lim_{n\to\infty} f_n = 0.$$