4 Construction of (Probability) Measures

Given: $\Omega \neq \emptyset$ and $\emptyset \neq \mathfrak{A} \subset \mathfrak{P}(\Omega)$.

Definition 1. $\mu : \mathfrak{A} \to \mathbb{R}_+ \cup \{\infty\}$ is called

(i) *additive* if:

$$A, B \in \mathfrak{A} \wedge A \cap B = \emptyset \wedge A \cup B \in \mathfrak{A} \quad \Rightarrow \quad \mu(A \cup B) = \mu(A) + \mu(B),$$

(ii) *σ-additive* if

$$A_1, A_2, \ldots \in \mathfrak{A} \text{ pairwise disjoint } \wedge \bigcup_{i=1}^{\infty} A_i \in \mathfrak{A} \quad \Rightarrow \quad \mu\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} \mu(A_i),$$

(iii) *content (on $\mathfrak{A}$)* if

$$\mathfrak{A} \text{ algebra} \quad \wedge \quad \mu \text{ additive} \quad \wedge \quad \mu(\emptyset) = 0,$$

(iv) *pre-measure (on $\mathfrak{A}$)* if

$$\mathfrak{A} \text{ semi-algebra} \quad \wedge \quad \mu\ \sigma\text{-additive} \quad \wedge \quad \mu(\emptyset) = 0,$$

(v) *measure (on $\mathfrak{A}$)* if

$$\mathfrak{A}\ \sigma\text{-algebra} \quad \wedge \quad \mu \text{ pre-measure},$$

(vi) *probability measure (on $\mathfrak{A}$)* if

$$\mu \text{ measure} \quad \wedge \quad \mu(\Omega) = 1.$$

Definition 2. $(\Omega, \mathfrak{A}, \mu)$ is called a

(i) *measure space*, if μ is a measure on the σ-algebra $\mathfrak{A}$ in Ω,

(ii) *probability space*, if μ is a probability measure on the σ-algebra $\mathfrak{A}$ in Ω.

Example 1.

(i) *k–dimensional Lebesgue pre-measure* λ_k, e.g., on cartesian products of intervals.

(ii) For any semi-algebra $\mathfrak{A}$ in Ω and $\omega \in \Omega$

$$\delta_\omega(A) = 1_A(\omega), \qquad A \in \mathfrak{A},$$

defines a pre-measure. If $\mathfrak{A}$ is a σ-algebra, then δ_ω is called the *Dirac measure* at the point ω.

More generally: take sequences $(\omega_n)_{n\in\mathbb{R}}$ in Ω and $(\alpha_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+$ such that $\sum_{n=1}^{\infty} \alpha_n = 1$. Then

$$\mu(A) = \sum_{n=1}^{\infty} \alpha_n \cdot 1_A(\omega_n), \qquad A \in \mathfrak{A},$$

defines a *discrete probability measure* on any σ-algebra $\mathfrak{A}$ in Ω. Note that $\mu = \sum_{n=1}^{\infty} \alpha_n \cdot \varepsilon_{\omega_n}$.

(iii) *Counting measure* on a σ-algebra $\mathfrak{A}$

$$\mu(A) = |A|, \qquad A \in \mathfrak{A}.$$

Uniform distribution in the case $|\Omega| < \infty$ and $\mathfrak{A} = \mathfrak{P}(\Omega)$

$$\mu(A) = \frac{|A|}{|\Omega|}, \qquad A \subset \Omega.$$

(iv) On the algebra $\mathfrak{A} = \{A \subset \Omega : A \text{ finite or } A^c \text{ finite}\}$ let

$$\mu(A) = \begin{cases} 0 & \text{if } |A| < \infty \\ \infty & \text{if } |A| = \infty. \end{cases}$$

Then μ is a content but not a pre-measure in general.

(v) For the semi-algebra of measurable rectangles in Example 3.1 and $A_i \subset \{0,1\}$

$$\mu(A_1 \times \cdots \times A_n \times \Omega_{n+1} \times \cdots) = \frac{|A_1 \times \ldots \times A_n|}{|\,\{0,1\}^n\,|}$$

is well defined and yields a pre-measure μ with $\mu\big(\{0,1\}^{\mathbb{N}}\big) = 1$.

Remark 1. For every content μ on $\mathfrak{A}$ and $A, B \in \mathfrak{A}$

(i) $A \subset B \Rightarrow \mu(A) \leq \mu(A \cap B) + \mu(A^c \cap B) = \mu(B)$ (*monotonicity*),

(ii) $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B \setminus A) + \mu(A \cap B) = \mu(A) + \mu(B)$,

(iii) $A \subset B \wedge \mu(A) < \infty \Rightarrow \mu(B \setminus A) = \mu(B) - \mu(A)$,

(iv) $\mu(A) < \infty \wedge \mu(B) < \infty \Rightarrow |\mu(A) - \mu(B)| \leq \mu(A \triangle B)$,

(v) $\mu(A \cup B) = \mu(A) + \mu(B \cap A^c) \leq \mu(A) + \mu(B)$ (*subadditivity*).

Theorem 1. Consider the following properties for a content μ on $\mathfrak{A}$:

(i) μ pre-measure,

(ii) $A_1, A_2, \ldots \in \mathfrak{A} \wedge \bigcup_{i=1}^\infty A_i \in \mathfrak{A} \Rightarrow \mu\big(\bigcup_{i=1}^\infty A_i\big) \leq \sum_{i=1}^\infty \mu(A_i)$ (*σ-subadditivity*),

(iii) $A_1, A_2, \ldots \in \mathfrak{A} \wedge A_n \uparrow A \in \mathfrak{A} \Rightarrow \lim_{n\to\infty} \mu(A_n) = \mu(A)$ (*σ-continuity from below*),

(iv) $A_1, A_2, \ldots \in \mathfrak{A} \wedge A_n \downarrow A \in \mathfrak{A} \wedge \mu(A_1) < \infty \Rightarrow \lim_{n\to\infty} \mu(A_n) = \mu(A)$ (*σ-continuity from above*),

(v) $A_1, A_2, \ldots \in \mathfrak{A} \wedge A_n \downarrow \emptyset \wedge \mu(A_1) < \infty \Rightarrow \lim_{n\to\infty} \mu(A_n) = 0$ (*σ-continuity at $\emptyset$*).

Then

$$\text{(i)} \Leftrightarrow \text{(ii)} \Leftrightarrow \text{(iii)} \Rightarrow \text{(iv)} \Leftrightarrow \text{(v)}.$$

If $\mu(\Omega) < \infty$, then (iii) $\Leftrightarrow$ (iv).

Proof. '(i) $\Rightarrow$ (ii)': Put $B_m = \bigcup_{i=1}^m A_i$ and $B_0 = \emptyset$. Then

$$\bigcup_{i=1}^\infty A_i = \bigcup_{m=1}^\infty (B_m \setminus B_{m-1})$$

with pairwise disjoint sets $B_m \setminus B_{m-1} \in \mathfrak{A}$. Clearly $B_m \setminus B_{m-1} \subset A_m$. Hence, by Remark 1.(i),

$$\mu\Big(\bigcup_{i=1}^\infty A_i\Big) = \sum_{m=1}^\infty \mu(B_m \setminus B_{m-1}) \leq \sum_{m=1}^\infty \mu(A_m).$$

'(ii) $\Rightarrow$ (i)': Let $A_1, A_2, \ldots \in \mathfrak{A}$ be pairwise disjoint with $\bigcup_{i=1}^\infty A_i \in \mathfrak{A}$. Then

$$\mu\Big(\bigcup_{i=1}^\infty A_i\Big) \geq \mu\Big(\bigcup_{i=1}^n A_i\Big) = \sum_{i=1}^n \mu(A_i),$$

and therefore

$$\sum_{i=1}^\infty \mu(A_i) \leq \mu\Big(\bigcup_{i=1}^\infty A_i\Big).$$

The reverse estimate holds by assumption.

'(i) $\Rightarrow$ (iii)': Put $A_0 = \emptyset$ and $B_m = A_m \setminus A_{m-1}$. Then

$$\mu\Big(\bigcup_{i=1}^\infty A_i\Big) = \sum_{m=1}^\infty \mu(B_m) = \lim_{n\to\infty} \sum_{m=1}^n \mu(B_m) = \lim_{n\to\infty} \mu\Big(\bigcup_{m=1}^n B_m\Big) = \lim_{n\to\infty} \mu(A_n).$$

'(iii) $\Rightarrow$ (i)': Let $A_1, A_2, \ldots \in \mathfrak{A}$ be pairwise disjoint with $\bigcup_{i=1}^\infty A_i \in \mathfrak{A}$, and put $B_m = \bigcup_{i=1}^m A_i$. Then $B_m \uparrow \bigcup_{i=1}^\infty A_i$ and

$$\mu\Big(\bigcup_{i=1}^\infty A_i\Big) = \lim_{m\to\infty} \mu(B_m) = \sum_{i=1}^\infty \mu(A_i).$$

'(iv) $\Rightarrow$ (v)' trivially holds.

'(v) $\Rightarrow$ (iv)': Use $B_n = A_n \setminus A \downarrow \emptyset$.

'(i)' $\Rightarrow$ (v)': Note that $\mu(A_1) = \sum_{i=1}^\infty \mu(A_i \setminus A_{i+1})$. Hence

$$0 = \lim_{k\to\infty} \sum_{i=k}^\infty \mu(A_i \setminus A_{i+1}) = \lim_{k\to\infty} \mu(A_k).$$

'(iv) $\wedge$ $\mu(\Omega) < \infty \Rightarrow$ (iii)': Clearly $A_n \uparrow A$ implies $A_n^c \downarrow A^c$. Thus

$$\mu(A) = \mu(\Omega) - \mu(A^c) = \lim_{n\to\infty} (\mu(\Omega) - \mu(A_n^c)) = \lim_{n\to\infty} \mu(A_n).$$

$\square$

Theorem 2 (Extension: semi-algebra $\rightsquigarrow$ algebra). For every semi-algebra $\mathfrak{A}$ and every additive mapping $\mu : \mathfrak{A} \to \mathbb{R}_+ \cup \{\infty\}$ with $\mu(\emptyset) = 0$

$$\underset{1}{\exists}\, \widehat{\mu} \text{ content on } \alpha(\mathfrak{A}) : \quad \widehat{\mu}|_{\mathfrak{A}} = \mu.$$

Moreover, if μ is σ-additive then $\widehat{\mu}$ is σ-additive, too.

Proof. We have $\alpha(\mathfrak{A}) = \mathfrak{A}^+$, see Lemma 1.1. Necessarily

$$\widehat{\mu}\Big(\bigcup_{i=1}^{n} A_i\Big) = \sum_{i=1}^{n} \mu(A_i) \tag{1}$$

for $A_1, \dots, A_n \in \mathfrak{A}$ pairwise disjoint. Use (1) to obtain a well-defined extension of μ onto $\alpha(\mathfrak{A})$. It easily follows that μ is additive or even σ-additive. □

Example 2. For the semi-algebra $\mathfrak{A}$ in Example 1.(v) $\alpha(\mathfrak{A})$ is the algebra of cylinder sets, and

$$\widehat{\mu}(A \times \Omega_{n+1} \times \cdots) = \frac{|A|}{|\{0,1\}^n|}, \qquad A \subset \{0,1\}^n .$$

Let μ be a pre–measure on $\mathfrak{A}$. The *outer measure* generated by μ is

$$\mu^*(A) := \inf\left\{ \sum_{i=1}^{\infty} \mu(A_i) \ : \ A_i \in \mathfrak{A}, A \subseteq \bigcup_{i=1} \infty A_i \right\} ,$$

It is straightforward that $\mu^*(\emptyset = 0)$ and that μ^* is monotone and σ–subadditive.

Theorem 3 (Extension: algebra $\rightsquigarrow$ σ-algebra, Carathéodory). For every pre-measure μ on an algebra $\mathfrak{A}$,

(a) the class

$$\mathfrak{A}_{\mu^*} := \Big\{ A \subseteq \Omega \ : \ \mu^*(B) = \mu^*(A \cap B) + \mu^*(A^c \cap B) \forall B \subseteq \Omega \Big\}$$

is a σ–algebra, and μ^* is a measure on $\mathfrak{A}_{\mu^*}$.

(b) $\mathfrak{A} \subseteq \mathfrak{A}_{\mu^*}$, and $\mu = \mu^*$ on $\mathfrak{A}$. In particular, there exists a measure μ^* on $\sigma(\mathfrak{A})$ extending μ.

Proof. We will start with part (b), i.e., we show that

(i) $\mu^*|_{\mathfrak{A}} = \mu$,

(ii) $\forall\, A \in \mathfrak{A}\ \forall\, B \in \mathfrak{P}(\Omega):\quad \mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c)$.

Ad (i): For $A \in \mathfrak{A}$

$$\mu^*(A) \leq \mu(A) + \sum_{i=2}^{\infty} \mu(\emptyset) = \mu(A),$$

and for $A_i \in \mathfrak{A}$ with $A \subset \bigcup_{i=1}^{\infty} A_i$

$$\mu(A) = \mu\Big(\bigcup_{i=1}^{\infty} (A_i \cap A)\Big) \leq \sum_{i=1}^{\infty} \mu(A_i \cap A) \leq \sum_{i=1}^{\infty} \mu(A_i)$$

follows from Theorem 1.(ii).

Ad (ii): '$\leq$' holds due to sub-additivity of μ^*; if

$$B \subseteq \bigcup_{i=1}^{\infty} A_i$$

with $A_i \in \mathfrak{A}$, then $A_i \cap A, A_i \cap A^c \in \mathfrak{A}$ and

$$B \cap A \subseteq \bigcup_{i=1}^{\infty} A_i \cap A, \qquad B \cap A^c \subseteq \bigcup_{i=1}^{\infty} A_i \cap A^c \,.$$

This directly implies '$\geq$'.

Now we prove (a); to this end, we claim first that

(iii) $\mathfrak{A}_{\mu^*}$ is $\cap$–closed, $\forall\, A_1, A_2 \in \mathfrak{A}_{\mu^*}\ \forall\, B \in \mathfrak{P}(\Omega):\quad \mu^*(B) = \mu^*(B \cap (A_1 \cap A_2)) + \mu^*(B \cap (A_1 \cap A_2)^c)$.

(iv) $\mathfrak{A}_{\mu^*}$ ^{c}–closed,

i.e., $\mathfrak{A}$ is an algebra.
Ad (iii): We have

$$\begin{aligned}\mu^*(B) &= \mu^*(B \cap A_1) + \mu^*(B \cap A_1^c)\\ &= \mu^*(B \cap A_1 \cap A_2) + \mu^*(B \cap A_1 \cap A_2^c) + \mu^*(B \cap A_1^c)\end{aligned}$$

and

$$\mu^*(B \cap (A_1 \cap A_2)^c) = \mu^*(B \cap A_1^c \cup B \cap A_2^c) = \mu^*(B \cap A_2^c \cap A_1) + \mu^*(B \cap A_1^c).$$

Ad (iv): Obvious.
Next we claim that μ^* is additive on $\mathfrak{A}^*$, and even more,

(v) $\forall\, A_1, A_2 \in \mathfrak{A}_{\mu^*}$ disjoint $\forall\, B \in \mathfrak{P}(\Omega):\quad \mu^*(B \cap (A_1 \cup A_2)) = \mu^*(B \cap A_1) + \mu^*(B \cap A_2)$.

In fact, since $A_1 \cap A_2 = \emptyset$,

$$\mu^*(B \cap (A_1 \cup A_2)) = \mu^*(B \cap A_1) + \mu^*(B \cap A_2 \cap A_1^c) = \mu^*(B \cap A_1) + \mu^*(B \cap A_2).$$

At last, we claim that $\mathfrak{A}^*$ is a Dynkin class and μ^* is σ–additive on $\mathfrak{A}^*$, i.e.,

(vi) $\forall\, A_1, A_2, \ldots \in \mathfrak{A}_{\mu^*}$ pairwise disjoint

$$\bigcup_{i=1}^{\infty} A_i \in \mathfrak{A}_{\mu^*} \quad \wedge \quad \mu^*\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} \mu^*(A_i).$$

Let $B \in \mathfrak{P}(\Omega)$. By (iv), (v), and monotonicity of μ^*

$$\begin{aligned}\mu^*(B) &= \mu^*\Big(B \cap \bigcup_{i=1}^n A_i\Big) + \mu^*\Big(B \cap \Big(\bigcup_{i=1}^n A_i\Big)^c\Big) \\ &\geq \sum_{i=1}^n \mu^*(B \cap A_i) + \mu^*\Big(B \cap \Big(\bigcup_{i=1}^\infty A_i\Big)^c\Big).\end{aligned}$$

Use σ-subadditivity of μ^* to get

$$\begin{aligned}\mu^*(B) &\geq \sum_{i=1}^\infty \mu^*(B \cap A_i) + \mu^*\Big(B \cap \Big(\bigcup_{i=1}^\infty A_i\Big)^c\Big) \\ &\geq \mu^*\Big(B \cap \bigcup_{i=1}^\infty A_i\Big) + \mu^*\Big(B \cap \Big(\bigcup_{i=1}^\infty A_i\Big)^c\Big) \\ &\geq \mu^*(B).\end{aligned}$$

Hence $\bigcup_{i=1}^\infty A_i \in \mathfrak{A}_{\mu^*}$. Take $B = \bigcup_{i=1}^\infty A_i$ to obtain σ-additivity of $\mu^*|_{\mathfrak{A}_{\mu^*}}$.
Conclusions:

- $\mathfrak{A}_{\mu^*}$ is a Dynkin class and $\cap$–closed ((iv), (vi)), and hence a σ-algebra, see Theorem 1.1.(ii),
- $\mathfrak{A} \subset \mathfrak{A}_{\mu^*}$ by (ii), hence $\sigma(\mathfrak{A}) \subset \mathfrak{A}_{\mu^*}$.
- $\mu^*|_{\mathfrak{A}_{\mu^*}}$ is a measure with $\mu^*|_{\mathfrak{A}} = \mu$, see (vi) and (i).

□

Remark 2. The extension from Theorem 3 is non-unique, in general. For instance, on $\Omega = \mathbb{R}$, the pre–measure

$$\mu(A) = \infty \cdot \#A = \begin{cases} 0 & \text{if } A = \emptyset \\ \infty & \text{otherwise} \end{cases}, \qquad A \in \alpha(\mathfrak{J}_1)$$

on the algebra generated by intervals (see Ex.1) has the extensions $\mu_1(A) = \#A$ (counting measure) and $\mu_2(A) = \infty \cdot \#A$ to $\mathfrak{B}$.

Definition 3. $\mu : \mathfrak{A} \to \mathbb{R}_+ \cup \{\infty\}$ is called

(i) σ-*finite*, if

$$\exists\, B_1, B_2, \ldots \in \mathfrak{A} \text{ pairwise disjoint} : \quad \Omega = \bigcup_{i=1}^\infty B_i \wedge \forall\, i \in \mathbb{N} : \mu(B_i) < \infty,$$

(ii) *finite*, if $\Omega \in \mathfrak{A}$ and $\mu(\Omega) < \infty$.

Theorem 4 (Uniqueness). $\mathfrak{A}_0$ be $\cap$–closed, μ_1, μ_2 be measures on $\mathfrak{A} = \sigma(\mathfrak{A}_0)$. If $\mu_1|_{\mathfrak{A}_0}$ is σ–finite and $\mu_1|_{\mathfrak{A}_0} = \mu_2|_{\mathfrak{A}_0}$, then $\mu_1 = \mu_2$.

Proof. Take B_i according to Definition 3, with $\mathfrak{A}_0$ instead of $\mathfrak{A}$, and put

$$\mathfrak{D}_i = \{A \in \mathfrak{A} : \mu_1(A \cap B_i) = \mu_2(A \cap B_i)\}.$$

Obviously, $\mathfrak{D}_i$ is a Dynkin class and $\mathfrak{A}_0 \subset \mathfrak{D}_i$. Theorem 1.2.(i) yields

$$\mathfrak{D}_i \subset \mathfrak{A} = \sigma(\mathfrak{A}_0) = \delta(\mathfrak{A}_0) \subset \mathfrak{D}_i.$$

Thus $\mathfrak{A} = \mathfrak{D}_i$ and for $A \in \mathfrak{A}$,

$$\mu_1(A) = \sum_{i=1}^{\infty} \mu_1(A \cap B_i) = \sum_{i=1}^{\infty} \mu_2(A \cap B_i) = \mu_2(A).$$

□

Corollary 1. For every semi-algebra $\mathfrak{A}$ and every pre-measure μ on $\mathfrak{A}$ that is σ-finite

$$\underset{1}{\exists}\, \mu^* \text{ measure on } \sigma(\mathfrak{A}) : \quad \mu^*|_{\mathfrak{A}} = \mu.$$

Proof. Use Theorems 2, 3, and 4. □

Remark 3. Applications of Corollary 1:

(i) For $\Omega = \mathbb{R}^k$ and the Lebesgue pre-measure λ_k on $\mathfrak{J}_k$ we get the Lebesgue measure on $\mathfrak{B}_k$. Notation for the latter: λ_k.

(ii) In Example 1.(v) there exists a uniquely determined probability measure P on $\bigotimes_{i=1}^{\infty} \mathfrak{P}(\{0,1\})$ such that

$$P(A_1 \times \cdots \times A_n \times \{0,1\} \times \ldots) = \frac{|A_1 \times \cdots \times A_n|}{|\{0,1\}^n|}$$

for $A_1, \ldots, A_n \subset \{0,1\}$. We will study the general construction of product measures in Section 8.