3 Product Spaces

Example 1. A stochastic model for coin tossing. For a single trial,

$$\Omega = \{0,1\}, \quad \mathfrak{A} = \mathfrak{P}(\Omega), \quad \forall\, \omega \in \Omega : P(\{\omega\}) = 1/2. \tag{1}$$

For n 'independent' trials, (1) serves as a building-block,

$$\Omega_i = \{0,1\}, \quad \mathfrak{A}_i = \mathfrak{P}(\Omega_i), \quad \forall\, \omega_i \in \Omega_i : P_i(\{\omega_i\}) = 1/2,$$

and we define

$$\Omega = \prod_{i=1}^{n} \Omega_i, \quad \mathfrak{A} = \mathfrak{P}(\Omega), \quad \forall\, A \in \mathfrak{A} : P(A) = \frac{|A|}{|\Omega|}.$$

Then

$$P(A_1 \times \cdots \times A_n) = P_1(A_1) \cdot \cdots \cdot P_n(A_n)$$

for all $A_i \in \mathfrak{A}_i$.
Question: How to model an infinite sequence of trials? To this end,

$$\Omega = \prod_{i=1}^{\infty} \Omega_i.$$

How to choose a σ-algebra $\mathfrak{A}$ in Ω and a probability measure P on $(\Omega, \mathfrak{A})$? A reasonable requirement is

$$\begin{aligned} &\forall\, n \in \mathbb{N}\ \forall\, A_i \in \mathfrak{A}_i : \\ &\qquad P(A_1 \times \cdots \times A_n \times \Omega_{n+1} \times \Omega_{n+2} \ldots) = P_1(A_1) \cdot \cdots \cdot P_n(A_n). \end{aligned} \tag{2}$$

Unfortunately,

$$\mathfrak{A} = \mathfrak{P}(\Omega)$$

is too large, since there exists no probability measure on $(\Omega, \mathfrak{P}(\Omega))$ such that (2) holds. The latter fact follows from a theorem by Banach and Kuratowski, which relies on the continuum hypothesis, see Dudley (2002, p. 526). On the other hand,

$$\mathfrak{A} = \{A_1 \times \cdots \times A_n \times \Omega_{n+1} \times \Omega_{n+2} \cdots : n \in \mathbb{N},\ A_i \in \mathfrak{A}_i \text{ for } i = 1, \ldots, n\} \tag{3}$$

is not a σ-algebra.

Given: a non-empty set I and measurable spaces $(\Omega_i, \mathfrak{A}_i)$ for $i \in I$. Put

$$Y = \bigcup_{i \in I} \Omega_i$$

and define

$$\prod_{i \in I} \Omega_i = \{\omega \in Y^I : \omega(i) \in \Omega_i \text{ for } i \in I\}.$$

Notation: $\omega = (\omega_i)_{i \in I}$ for $\omega \in \prod_{i \in I} \Omega_i$. Moreover, let

$$\mathfrak{P}_0(I) = \{J \subset I : J \text{ non-empty, finite}\}.$$

The following definition is motivated by (3).

Definition 1.

(i) *Measurable rectangle*

$$A = \prod_{j \in J} A_j \times \prod_{i \in I \setminus J} \Omega_i$$

with $J \in \mathfrak{P}_0(I)$ and $A_j \in \mathfrak{A}_j$ for $j \in J$. Notation: $\mathfrak{R}$ class of measurable rectangles.

(ii) *Product (measurable) space* $(\Omega, \mathfrak{A})$ with *components* $(\Omega_i, \mathfrak{A}_i)$, $i \in I$,

$$\Omega = \prod_{i \in I} \Omega_i, \qquad \mathfrak{A} = \sigma(\mathfrak{R}).$$

Notation: $\mathfrak{A} = \bigotimes_{i \in I} \mathfrak{A}_i$, *product σ-algebra*.

Remark 1. The class $\mathfrak{R}$ is a semi-algebra, but not an algebra in general. See Übung 2.3.

Example 2. Obviously, (2) only makes sense if $\mathfrak{A}$ contains the product σ-algebra $\bigotimes_{i=1}^{n} \mathfrak{A}_i$. We will show that there exists a uniquely determined probability measure P on the product space $(\prod_{i=1}^{\infty}\{0,1\}, \bigotimes_{i=1}^{\infty} \mathfrak{P}(\{0,1\}))$ that satisfies (2), see Remark 4.3.(ii). The corresponding probability space yields a stochastic model for the simple case of gambling, which was mentioned in the introductory Example I.2.

We study several classes of mappings or subsets that generate the product σ-algebra. Moreover, we characterize measurability of mappings that take values in a product space.

Put $\Omega = \prod_{i \in I} \Omega_i$. For any $\emptyset \neq S \subset I$ let

$$\pi_S^I : \Omega \to \prod_{i \in S} \Omega_i, \quad (\omega_i)_{i \in I} \mapsto (\omega_i)_{i \in S}$$

denote the *projection* of Ω onto $\prod_{i \in S} \Omega_i$ (restriction of mappings ω). In particular, for $i \in I$ the i-th projection is given by $\pi_{\{i\}}^I$. Sometimes we simply write π_S instead of π_S^I and π_i instead of $\pi_{\{i\}}$.

Theorem 1.

(i) $\bigotimes_{i\in I}\mathfrak{A}_i = \sigma(\{\pi_i : i \in I\})$.

(ii) $\forall\, i \in I : \mathfrak{A}_i = \sigma(\mathfrak{E}_i) \quad \Rightarrow \quad \bigotimes_{i\in I}\mathfrak{A}_i = \sigma\left(\bigcup_{i\in I}\pi_i^{-1}(\mathfrak{E}_i)\right)$.

Proof. Ad (i), '$\supset$': We show that every projection $\pi_i : \Omega \to \Omega_i$ is $\left(\bigotimes_{i\in I}\mathfrak{A}_i\right)$-$\mathfrak{A}_i$-measurable. For $A_i \in \mathfrak{A}_i$

$$\pi_i^{-1}(A_i) = A_i \times \prod_{i\in I\setminus\{i\}}\Omega_i \in \mathfrak{R}.$$

Ad (i), '$\subset$': We show that $\mathfrak{R} \subset \sigma(\{\pi_i : i \in I\})$. For $J \in \mathfrak{P}_0(I)$ and $A_j \in \mathfrak{A}_j$ with $j \in J$

$$\prod_{j\in J}A_j \times \prod_{i\in I\setminus J}\Omega_i = \bigcap_{j\in J}\pi_j^{-1}(A_j).$$

Ad (ii): By Lemma 2.1 and (i)

$$\bigotimes_{i\in I}\mathfrak{A}_i = \sigma\Big(\bigcup_{i\in I}\pi_i^{-1}(\mathfrak{A}_i)\Big) = \sigma\Big(\bigcup_{i\in I}\sigma(\pi_i^{-1}(\mathfrak{E}_i))\Big) = \sigma\Big(\bigcup_{i\in I}\pi_i^{-1}(\mathfrak{E}_{\mathrm{i}})\Big).$$

$\square$

Corollary 1.

(i) For every measurable space $(\widetilde{\Omega}, \widetilde{\mathfrak{A}})$ and every mapping $g : \widetilde{\Omega} \to \Omega$

$$g \text{ is } \widetilde{\mathfrak{A}}\text{-}\bigotimes_{i\in I}\mathfrak{A}_i\text{-measurable} \quad \Leftrightarrow \quad \forall\, i \in I : \pi_i \circ g \text{ is } \widetilde{\mathfrak{A}}\text{-}\mathfrak{A}_i\text{-measurable.}$$

(ii) For every $\emptyset \neq S \subset I$ the projection π_S^I is $\bigotimes_{i\in I}\mathfrak{A}_i$-$\bigotimes_{i\in S}\mathfrak{A}_i$-measurable.

Proof. Ad (i): Follows immediately from Theorem 2.3 and Theorem 1.(i).
Ad (ii): Note that $\pi_{\{i\}}^S \circ \pi_S^I = \pi_i^I$ and use (i). $\square$

Remark 2. From Theorem 1.(i) and Corollary 1 we get

$$\bigotimes_{i\in I}\mathfrak{A}_i = \sigma(\{\pi_S^I : S \in \mathfrak{P}_0(I)\}).$$

The sets

$$\left(\pi_S^I\right)^{-1}(B) = B \times \Big(\prod_{i\in I\setminus S}\Omega_i\Big)$$

with $S \in \mathfrak{P}_0(I)$ and $B \in \bigotimes_{i\in S}\mathfrak{A}_i$ are called *cylinder sets*. Notation: $\mathfrak{C}$ class of cylinder sets. The class $\mathfrak{C}$ is an algebra in Ω, but not a σ-algebra in general. Moreover,

$$\mathfrak{R} \subset \alpha(\mathfrak{R}) \subset \mathfrak{C} \subset \sigma(\mathfrak{R}),$$

where equality does not hold in general.

Every product measurable set is countably determined in the following sense.

Theorem 2. For every $A \in \otimes_{i\in I}\mathfrak{A}_i$ there exists a non-empty countable set $S \subset I$ and a set $B \in \otimes_{i\in S}\mathfrak{A}_i$ such that

$$A = \left(\pi_S^I\right)^{-1}(B).$$

Proof. Put

$$\widetilde{\mathfrak{A}} = \left\{A \in \bigotimes_{i\in I}\mathfrak{A}_i : \exists\, S \subset I \text{ non-empty, countable } \exists\, B \in \bigotimes_{i\in S}\mathfrak{A}_i : A = \left(\pi_S^I\right)^{-1}(B)\right\}.$$

By definition, $\widetilde{\mathfrak{A}}$ contains every cylinder set and $\widetilde{\mathfrak{A}} \subset \bigotimes_{i\in I}\mathfrak{A}_i$. It remains to show that $\widetilde{\mathfrak{A}}$ is a σ-algebra. Obviously, $\Omega \in \widetilde{\mathfrak{A}}$, and if $A = (\pi_S^I)^{-1}(B)$, $A^c = (\pi_S^I)^{-1}(B^c)$. Finally, if $A_n = (\pi_{S_n}^I)^{-1}(B_n)$, we define $S = \bigcup_n S_n$ and $\widetilde{B}_n = (\pi_{S_n}^S)^{-1}(B_n) = B_n \times \prod_{i\in S\setminus B_n} \in \bigotimes_{i\in S}\mathfrak{A}_i$ (see Corollary 1, (ii)); then

$$\bigcap_n A_n = \bigcap_n (\pi_S^I)^{-1}(\widetilde{B}_n) = ((\pi^I)_S)^{-1}\left(\bigcap_n \widetilde{B}_n\right),$$

hence $\bigcap_n A_n \in \widetilde{\mathfrak{A}}$. □

Now we study products of Borel-σ-algebras.

Theorem 3.

$$\mathfrak{B}_k = \bigotimes_{i=1}^k \mathfrak{B}, \qquad \overline{\mathfrak{B}}_k = \bigotimes_{i=1}^k \overline{\mathfrak{B}}.$$

Proof. ByRemarkrefch2s1.refch2r5,

$$B_k = \sigma\Big(\Big\{\prod_{i=1}^k]-\infty, a_i] : a_i \in \mathbb{R} \text{ for } i = 1, \ldots, k\Big\}\Big) \subset \bigotimes_{i=1}^k \mathfrak{B}.$$

On the other hand, $\pi_i : \mathbb{R}^k \to \mathbb{R}$ is continuous, hence it remains to apply Corollary 2.1 and Theorem 1.(i). Analogously, $\overline{\mathfrak{B}}_k = \bigotimes_{i=1}^k \overline{\mathfrak{B}}$ follows. □

Remark 3. Consider a measurable space $(\widetilde{\Omega}, \widetilde{\mathfrak{A}})$ and a mapping

$$f = (f_1, \ldots, f_k) : \widetilde{\Omega} \to \overline{\mathbb{R}}^k.$$

Then, according to Theorem 3, f is $\widetilde{\mathfrak{A}}$-$\overline{\mathfrak{B}}_k$-measurable iff all functions f_i are $\widetilde{\mathfrak{A}}$-$\overline{\mathfrak{B}}$-measurable.