2 Measurable Mappings

Definition 1. $(\Omega, \mathfrak{A})$ is called *measurable space* iff $\Omega \neq \emptyset$ and $\mathfrak{A}$ is a σ-algebra in Ω. Elements $A \in \mathfrak{A}$ are called $(\mathfrak{A}-)$*measurable sets*.

In the sequel, $(\Omega_i, \mathfrak{A}_i)$ are measurable spaces for $i = 1, 2, 3$.

Remark 1. Let $f : \Omega_1 \to \Omega_2$. For $B \in \mathfrak{A}_2$, we set in short

$$\{f \in B\} = f^{-1}(B) = \{\omega \in \Omega_1 \ : \ f(\omega) \in B\} \subset \Omega_1$$

(i) $f^{-1}(\mathfrak{A}_2) = \{f^{-1}(A) : A \in \mathfrak{A}_2\}$ is a σ-algebra in Ω_1.

(ii) $\{A \subset \Omega_2 : f^{-1}(A) \in \mathfrak{A}_1\}$ is a σ-algebra in Ω_2.

Definition 2. $f : \Omega_1 \to \Omega_2$ is $\mathfrak{A}_1$-$\mathfrak{A}_2$-*measurable* iff $f^{-1}(\mathfrak{A}_2) \subset \mathfrak{A}_1$. i.e., iff for all $A \in \mathfrak{A}_2$ we have $\{f \in A\} \in \mathfrak{A}_1$.

How can we prove measurability of a given mapping?

Theorem 1. If $f : \Omega_1 \to \Omega_2$ is $\mathfrak{A}_1$-$\mathfrak{A}_2$-measurable and $g : \Omega_2 \to \Omega_3$ is $\mathfrak{A}_2$-$\mathfrak{A}_3$-measurable, then $g \circ f : \Omega_1 \to \Omega_3$ is $\mathfrak{A}_1$-$\mathfrak{A}_3$-measurable.

Proof. (Compare Bemerkung 5.4,(i), Analysis IV)

$$(g \circ f)^{-1}(\mathfrak{A}_3) = f^{-1}(g^{-1}(\mathfrak{A}_3)) \subset f^{-1}(\mathfrak{A}_2) \subset \mathfrak{A}_1 \, .$$

□

Lemma 1. For $f : \Omega_1 \to \Omega_2$ and $\mathfrak{E} \subset \mathfrak{P}(\Omega_2)$

$$f^{-1}(\sigma(\mathfrak{E})) = \sigma(f^{-1}(\mathfrak{E})).$$

Proof. By $f^{-1}(\mathfrak{E}) \subset f^{-1}(\sigma(\mathfrak{E}))$ and Remark 1.(i) we get $\sigma(f^{-1}(\mathfrak{E})) \subset f^{-1}(\sigma(\mathfrak{E}))$.
Let $\mathfrak{F} = \{A \subset \Omega_2 : f^{-1}(A) \in \sigma(f^{-1}(\mathfrak{E}))\}$. Then $\mathfrak{E} \subset \mathfrak{F}$ and $\mathfrak{F}$ is a σ-algebra, see Remark 1.(ii). Thus we get $\sigma(\mathfrak{E}) \subset \mathfrak{F}$, i.e., $f^{-1}(\sigma(\mathfrak{E})) \subset \sigma(f^{-1}(\mathfrak{E}))$. □

Theorem 2. If $\mathfrak{A}_2 = \sigma(\mathfrak{E})$ with $\mathfrak{E} \subset \mathfrak{P}(\Omega_2)$, then

$$f \text{ is } \mathfrak{A}_1\text{-}\mathfrak{A}_2\text{-measurable} \quad \Leftrightarrow \quad f^{-1}(\mathfrak{E}) \subset \mathfrak{A}_1 \, .$$

Proof. (compare Lemma 5.2, Analysis IV) '⇒' is trivial,
'⇐':Assume that $f^{-1}(\mathfrak{E}) \subset \mathfrak{A}_1$. By Lemma 1,

$$f^{-1}(\mathfrak{A}_2) = f^{-1}(\sigma(\mathfrak{E})) = \sigma(f^{-1}(\mathfrak{E})) \subset \sigma(\mathfrak{A}_1) = \mathfrak{A}_1.$$

□

Corollary 1. Let $(\Omega_i, \mathfrak{G}_i)$ be topological spaces. Then every continuous $f : \Omega_1 \to \Omega_2$ is $\mathfrak{B}(\Omega_1)$-$\mathfrak{B}(\Omega_2)$-measurable.

Proof. (Compare Korollar 5.3, Analysis IV) For continuous f we have

$$f^{-1}(\mathfrak{G}_2) \subset \mathfrak{G}_1 \subset \sigma(\mathfrak{G}_1) = \mathfrak{B}(\Omega_1).$$

Theorem 2 shows the claim. □

Given: measurable spaces $(\Omega_i, \mathfrak{A}_i)$ for $i \in I \neq \emptyset$, mappings $f_i : \Omega \to \Omega_i$ for $i \in I$ and some non-empty set Ω.

Definition 3. The *σ-algebra generated by $(f_i)_{i \in I}$ (and $(\mathfrak{A}_i)_{i \in I}$)*

$$\sigma(\{f_i : i \in I\}) = \sigma\Big(\bigcup_{i \in I} f_i^{-1}(\mathfrak{A}_i)\Big).$$

Moreover, set $\sigma(f) = \sigma(\{f\})$.

Remark 2. $\sigma(\{f_i : i \in I\})$ is the smallest σ-algebra $\mathfrak{A}$ in Ω such that all mappings f_i are $\mathfrak{A}$-$\mathfrak{A}_i$-measurable.

Theorem 3. For every measurable space $(\widetilde{\Omega}, \widetilde{\mathfrak{A}})$ and every mapping $g : \widetilde{\Omega} \to \Omega$,

$$g \text{ is } \widetilde{\mathfrak{A}}\text{-}\sigma(\{f_i : i \in I\})\text{-measurable} \quad \Leftrightarrow \quad \forall\, i \in I : f_i \circ g \text{ is } \widetilde{\mathfrak{A}}\text{-}\mathfrak{A}_i\text{-measurable}.$$

Proof. Use Lemma 1 to obtain

$$g^{-1}(\sigma(\{f_i : i \in I\})) = \sigma\Big(g^{-1}\Big(\bigcup_{i \in I} f_i^{-1}(\mathfrak{A}_i)\Big)\Big) = \sigma\Big(\bigcup_{i \in I} (f_i \circ g)^{-1}(\mathfrak{A}_i)\Big).$$

Therefore

$$g^{-1}\left(\sigma(\{f_i : i \in I\})\right) \subset \widetilde{\mathfrak{A}} \quad \Leftrightarrow \quad \forall\, i \in I : f_i \circ g_i \text{ is } \widetilde{\mathfrak{A}}\text{-}\mathfrak{A}_i\text{-measurable}.$$

□

Now we turn to the particular case of functions with values in $\mathbb{R}$ or $\overline{\mathbb{R}}$, and we consider the Borel σ-algebra in $\mathbb{R}$ or $\overline{\mathbb{R}}$, respectively. For any measurable space $(\Omega, \mathfrak{A})$ we use the following notation

$$\begin{aligned}
\mathfrak{Z}(\Omega, \mathfrak{A}) &= \{f : \Omega \to \mathbb{R} : f \text{ is } \mathfrak{A}\text{-}\mathfrak{B}\text{-measurable}\}, \\
\mathfrak{Z}_+(\Omega, \mathfrak{A}) &= \{f \in \mathfrak{Z}(\Omega, \mathfrak{A}) : f \geq 0\}, \\
\overline{\mathfrak{Z}}(\Omega, \mathfrak{A}) &= \left\{f : \Omega \to \overline{\mathbb{R}} : f \text{ is } \mathfrak{A}\text{-}\overline{\mathfrak{B}}\text{-measurable}\right\}, \\
\overline{\mathfrak{Z}}_+(\Omega, \mathfrak{A}) &= \left\{f \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A}) : f \geq 0\right\}.
\end{aligned}$$

Every function $f : \Omega \to \mathbb{R}$ may also be considered as a function with values in $\overline{\mathbb{R}}$, and in this case $f \in \mathfrak{Z}(\Omega, \mathfrak{A})$ iff $f \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$.

Corollary 2. For $\prec \in \{\leq, <, \geq, >\}$ and $f : \Omega \to \overline{\mathbb{R}}$,

$$f \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A}) \quad \Leftrightarrow \quad \forall\, a \in \mathbb{R} : \{f \prec a\} \in \mathfrak{A}.$$

Proof. (Compare Satz 5.6, Bem.5.7, Analysis IV) For instance, $\overline{\mathfrak{B}} = \sigma(\{[-\infty, a] : a \in \mathbb{R}\})$ and

$$\{f \le a\} = f^{-1}([-\infty, a])$$

and $\overline{\mathfrak{B}} = \sigma(\{[-\infty, a] : a \in \mathbb{R}\})$, see Remark 1.6. It remains to apply Theorem 2. □

Theorem 4. For $f, g \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$ and $\prec \in \{\le, <, \ge, >, =, \neq\}$,

$$\{\omega \in \Omega : f(\omega) \prec g(\omega)\} \in \mathfrak{A}.$$

Proof. For instance, Corollary 2 yields

$$\begin{aligned}\{\omega \in \Omega : f(\omega) < g(\omega)\} &= \bigcup_{q \in \mathbb{Q}} \{f < q < g\} \\ &= \bigcup_{q \in \mathbb{Q}} (\{f < q\} \cap \{g > q\}) \in \mathfrak{A}.\end{aligned}$$

□

Theorem 5. For every sequence $f_1, f_2, \ldots \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$,

(i) $\inf_{n \in \mathbb{N}} f_n,\ \sup_{n \in \mathbb{N}} f_n \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$,

(ii) $\liminf_{n \to \infty} f_n,\ \limsup_{n \to \infty} f_n \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$,

(iii) if $(f_n)_{n \in \mathbb{N}}$ converges at every point $\omega \in \Omega$, then $\lim_{n \to \infty} f_n \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$.

Proof. (Compare Satz 5.8, 5.9, Analysis IV) For $a \in \mathbb{R}$

$$\left\{\inf_{n \in \mathbb{N}} f_n < a\right\} = \bigcup_{n \in \mathbb{N}} \{f_n < a\}, \quad \left\{\sup_{n \in \mathbb{N}} f_n \le a\right\} = \bigcap_{n \in \mathbb{N}} \{f_n \le a\}.$$

Hence, Corollary 2 yields (i). Since

$$\limsup_{n \to \infty} f_n = \inf_{m \in \mathbb{N}} \sup_{n \ge m} f_n, \quad \liminf_{n \to \infty} f_n = \sup_{m \in \mathbb{N}} \inf_{n \ge m} f_n,$$

we obtain (ii) from (i). Finally, (iii) follows from (ii). □

By

$$f^+ = \max(0, f), \quad f^- = \max(0, -f)$$

we denote the positive part and the negative part, respectively, of $f : \Omega \to \overline{\mathbb{R}}$.

Remark 3. For $f \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$ we have $f^+, f^-, |f| \in \overline{\mathfrak{Z}}_+(\Omega, \mathfrak{A})$.

Theorem 6. For $f, g \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$,

$$f \pm g,\ f \cdot g,\ f/g \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A}),$$

provided that these functions are well defined.

Proof. (Compare Folgerung 5.5, Analysis IV) The proof is again based on Corollary 2. For simplicity we only consider the case that f and g are real-valued. Clearly $g \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$ implies $-g \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$, too. Furthermore, for every $a \in \mathbb{R}$,

$$\{f + g < a\} = \bigcup_{q \in \mathbb{Q}} \{f < q\} \cap \{g < a - q\},$$

and therefore $f \pm g \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$. Clearly $f \cdot g \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$ if f is constant. Moreover, $x \mapsto x^2$ defines a $\mathfrak{B}$-$\mathfrak{B}$-measurable function, see Corollary 1, and

$$f \cdot g = 1/4 \cdot \big((f + g)^2 - (f - g)^2\big)$$

We apply Theorem 1 to obtain $f \cdot g \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$ in general. Finally, it is easy to show that $g \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$ implies $1/g \in \overline{\mathfrak{Z}}(\Omega, \mathfrak{A})$. □

Definition 4. $f \in \mathfrak{Z}(\Omega, \mathfrak{A})$ is called *simple function* if $|f(\Omega)| < \infty$. Put

$$\Sigma(\Omega, \mathfrak{A}) = \{f \in \mathfrak{Z}(\Omega, \mathfrak{A}) : f \text{ simple}\},$$
$$\Sigma_+(\Omega, \mathfrak{A}) = \{f \in \Sigma(\Omega, \mathfrak{A}) : f \geq 0\}.$$

Remark 4. $f \in \Sigma(\Omega, \mathfrak{A})$ iff

$$f = \sum_{i=1}^{n} \alpha_i \cdot 1_{A_i}$$

with $\alpha_1, \dots \alpha_n \in \mathbb{R}$ pairwise different and $A_1, \dots, A_n \in \mathfrak{A}$ pairwise disjoint such that $\bigcup_{i=1}^{n} A_i = \Omega$.

Theorem 7. (Compare Theorem 5.11, Analysis IV) For every (bounded) function $f \in \overline{\mathfrak{Z}}_+(\Omega, \mathfrak{A})$ there exists a sequence $f_1, f_2, \dots \in \Sigma_+(\Omega, \mathfrak{A})$ such that $f_n \uparrow f$ (with uniform convergence).

Proof. Let $n \in N$ and put

$$f_n = \sum_{k=1}^{n \cdot 2^n} \frac{k-1}{2^n} \cdot 1_{A_{n,k}} + n \cdot 1_{B_n}$$

where

$$A_{n,k} = \{(k-1)/(2^n) \leq f < k/(2^n)\}, \quad B_n = \{f \geq n\}.$$

□

Now we consider a mapping $T : \Omega_1 \to \Omega_2$ and a σ-algebra $\mathfrak{A}_2$ in Ω_2. We characterize measurability of functions with respect to $\sigma(T) = T^{-1}(\mathfrak{A}_2)$.

Theorem 8 (Factorization Lemma). For every function $f : \Omega_1 \to \overline{\mathbb{R}}$

$$f \in \overline{\mathfrak{Z}}(\Omega_1, \sigma(T)) \quad \Leftrightarrow \quad \exists\, g \in \overline{\mathfrak{Z}}(\Omega_2, \mathfrak{A}_2) : f = g \circ T.$$

Proof. '$\Leftarrow$' is trivially satisfied. '$\Rightarrow$': First, assume that $f \in \Sigma_+(\Omega_1, \sigma(T))$, i.e.,

$$f = \sum_{i=1}^{n} \alpha_i \cdot 1_{A_i}$$

with pairwise disjoint sets $A_1, \ldots, A_n \in \sigma(T)$. Take pairwise disjoint sets $B_1, \ldots, B_n \in \mathfrak{A}_2$ such that $A_i = T^{-1}(B_i)$ and put

$$g = \sum_{i=1}^{n} \alpha_i \cdot 1_{B_i}.$$

Clearly $f = g \circ T$ and $g \in \overline{\mathfrak{Z}}(\Omega_2, \mathfrak{A}_2)$.
Now, assume that $f \in \overline{\mathfrak{Z}}_+(\Omega_1, \sigma(T))$. Take a sequence $(f_n)_{n\in\mathbb{N}}$ in $\Sigma_+(\Omega_1, \sigma(T))$ according to Theorem 7. We already know that $f_n = g_n \circ T$ for suitable $g_n \in \overline{\mathfrak{Z}}(\Omega_2, \mathfrak{A}_2)$. Hence

$$f = \sup_n f_n = \sup_n (g_n \circ T) = (\sup_n g_n) \circ T = g \circ T$$

where $g = \sup_n g_n \in \overline{\mathfrak{Z}}(\Omega_2, \mathfrak{A}_2)$.
In the general case, we already know that

$$f^+ = g_1 \circ T, \quad f^- = g_2 \circ T$$

for suitable $g_1, g_2 \in \overline{\mathfrak{Z}}(\Omega_2, \mathfrak{A}_2)$. Put

$$C = \{g_1 = g_2 = \infty\} \in \mathfrak{A}_2,$$

and observe that $T(\Omega_1) \cap C = \emptyset$ since $f = f^+ - f^-$. We conclude that $f = g \circ T$ where

$$g = g_1 \cdot 1_D - g_2 \cdot 1_D \in \overline{\mathfrak{Z}}(\Omega_2, \mathfrak{A}_2)$$

with $D = C^c$. □

Our method of proof for Theorem 8 is sometimes called *algebraic induction.*