Chapter II

Measure and Integral

1 Classes of Sets

Given: a non-empty set Ω and a class $\mathfrak{A} \subset \mathfrak{P}(\Omega)$ of subsets. Put

$$\mathfrak{A}^+ = \Big\{ \bigcup_{i=1}^n A_i : n \in \mathbb{N} \wedge A_1, \ldots, A_n \in \mathfrak{A} \text{ pairwise disjoint} \Big\}.$$

Definition 1.

(i) $\mathfrak{A}$ *closed w.r.t. intersections* or $\cap$*–closed* iff $A, B \in \mathfrak{A} \Rightarrow A \cap B \in \mathfrak{A}$.

(ii) $\mathfrak{A}$ *closed w.r.t. unions* or $\cup$*–closed* iff $A, B \in \mathfrak{A} \Rightarrow A \cup B \in \mathfrak{A}$.

(iii) $\mathfrak{A}$ *closed w.r.t. complements* or ^{c}*–closed* iff $A \in \mathfrak{A} \Rightarrow A^c := \Omega \setminus A \in \mathfrak{A}$.

(iv) $\mathfrak{A}$ *semi-algebra (in* Ω*)* if

  (a) $\Omega \in \mathfrak{A}$,
  (b) $\mathfrak{A}$ $\cap$–closed,
  (c) $A \in \mathfrak{A} \Rightarrow A^c \in \mathfrak{A}^+$.

(v) $\mathfrak{A}$ *algebra (in* Ω*)* if

  (a) $\Omega \in \mathfrak{A}$,
  (b) $\mathfrak{A}$ $\cap$–closed,
  (c) $\mathfrak{A}$ ^{c}–closed.

(vi) $\mathfrak{A}$ σ*-algebra (in* Ω*)* if

  (a) $\Omega \in \mathfrak{A}$,
  (b) $A_1, A_2, \ldots \in \mathfrak{A} \Rightarrow \bigcup_{n=1}^\infty A_n \in \mathfrak{A}$,
  (c) $\mathfrak{A}$ ^{c}–closed.

Remark 1. Let $\mathfrak{A}$ denote a σ-algebra in Ω. Recall that a *probability measure* P on $(\Omega, \mathfrak{A})$ is a mapping

$$P : \mathfrak{A} \to [0, 1]$$

such that $P(\Omega) = 1$ and

$$A_1, A_2, \ldots \in \mathfrak{A} \text{ pairwise disjoint} \quad \Rightarrow \quad P\Big(\bigcup_{i=1}^\infty A_i\Big) = \sum_{i=1}^\infty P(A_i).$$

Moreover, $(\Omega, \mathfrak{A}, P)$ is called a *probability space*, and $P(A)$ is the *probability* of the *event* $A \in \mathfrak{A}$.

Remark 2.

(i) $\mathfrak{A}$ σ-algebra $\Rightarrow$ $\mathfrak{A}$ algebra $\Rightarrow$ $\mathfrak{A}$ semi-algebra.

(ii) $\mathfrak{A}$ closed w.r.t. intersections $\Rightarrow \mathfrak{A}^+$ closed w.r.t. intersections.

(iii) $\mathfrak{A}$ algebra and $A_1, A_2 \in \mathfrak{A} \Rightarrow A_1 \cup A_2,\ A_1 \setminus A_2,\ A_1 \vartriangle A_2 \in \mathfrak{A}$.

(iv) $\mathfrak{A}$ σ-algebra and $A_1, A_2, \dots \in \mathfrak{A} \Rightarrow \bigcap_{n=1}^{\infty} A_n \in \mathfrak{A}$.

Example 1.

(i) Let $\Omega = \mathbb{R}$ and consider the class of intervals

$$\mathfrak{A} = \{]a,b] : a,b \in \mathbb{R} \wedge a < b\} \cup \{]-\infty, b] : b \in \mathbb{R}\} \cup \{]a, \infty[: a \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}.$$

Then $\mathfrak{A}$ is a semi-algebra, but not an algebra.

(ii) $\{A \in \mathfrak{P}(\Omega) : A \text{ finite or } A^c \text{ finite}\}$ is an algebra, but not a σ-algebra in general.

(iii) $\{A \in \mathfrak{P}(\Omega) : A \text{ countable or } A^c \text{ countable}\}$ is a σ-algebra.

(iv) $\mathfrak{P}(\Omega)$ is the largest σ-algebra in Ω, $\{\emptyset, \Omega\}$ is the smallest σ-algebra in Ω.

Definition 2.

(i) $\mathfrak{A}$ *monotone class (in Ω)* if

(a) $A_1, A_2, \ldots \in \mathfrak{A} \wedge A_n \uparrow A$[1] $\Rightarrow A \in \mathfrak{A}$,

(b) $A_1, A_2, \ldots \in \mathfrak{A} \wedge A_n \downarrow A$[2] $\Rightarrow A \in \mathfrak{A}$.

(ii) $\mathfrak{A}$ *Dynkin class (in Ω)* if

(a) $\Omega \in \mathfrak{A}$,

(b) $A_1, A_2 \in \mathfrak{A} \wedge A_1 \subset A_2 \Rightarrow A_2 \setminus A_1 \in \mathfrak{A}$,

(c) $A_1, A_2, \ldots \in \mathfrak{A}$ *pairwise disjoint* $\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathfrak{A}$.

Remark 3. $\mathfrak{A}$ σ-algebra $\Rightarrow$ $\mathfrak{A}$ monotone class and Dynkin class.

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[1] I.e., $A_n \subseteq A_{n+1}$ for all n and $A = \bigcup_n A_n$

[2] I.e., $A_{n+1} \subseteq A_n$ for all n and $A = \bigcap_n A_n$

Theorem 1.

(i) For every algebra $\mathfrak{A}$

$$\mathfrak{A}\ \sigma\text{-algebra} \quad \Leftrightarrow \quad \mathfrak{A} \text{ monotone class.}$$

(ii) For every Dynkin class $\mathfrak{A}$

$$\mathfrak{A}\ \sigma\text{-algebra} \quad \Leftrightarrow \quad \mathfrak{A} \text{ closed w.r.t. intersections.}$$

Proof. Ad (i), '$\Leftarrow$': Let $A_1, A_2, \ldots \in \mathfrak{A}$ and put $B_m = \bigcup_{n=1}^m A_n$ and $B = \bigcup_{n=1}^\infty A_n$. Then $B_m \uparrow B$. Furthermore, $B_m \in \mathfrak{A}$ since $\mathfrak{A}$ is an algebra. Thus $B \in \mathfrak{A}$ since $\mathfrak{A}$ is a monotone class.

Ad (ii), '$\Leftarrow$': For $A \in \mathfrak{A}$ we have $A^c = \Omega \setminus A \in \mathfrak{A}$ since $\mathfrak{A}$ is a Dynkin class. For $A, B \in \mathfrak{A}$ we have

$$A \cup B = A \cup (B \setminus (A \cap B)) \in \mathfrak{A}$$

since $\mathfrak{A}$ is also closed w.r.t. intersections. Thus, for $A_1, A_2, \ldots \in \mathfrak{A}$ and B_m as previously we get $B_m \in \mathfrak{A}$ and

$$\bigcup_{n=1}^\infty A_n = \bigcup_{m=1}^\infty (B_m \setminus B_{m-1}) \in \mathfrak{A},$$

where $B_0 = \emptyset$. □

Remark 4. Consider σ-algebras (algebras, monotone classes, Dynkin classes) $\mathfrak{A}_i$ for $i \in I \neq \emptyset$. Then $\bigcap_{i \in I} \mathfrak{A}_i$ is a σ-algebra (algebra, monotone class, Dynkin class), too.

Given: a class $\mathfrak{E} \subset \mathfrak{P}(\Omega)$.

Definition 3. The *σ-algebra generated by* $\mathfrak{E}$

$$\sigma(\mathfrak{E}) = \bigcap \{\mathfrak{A} : \mathfrak{A}\ \sigma\text{-algebra in } \Omega \wedge \mathfrak{E} \subset \mathfrak{A}\}.$$

Analogously, $\alpha(\mathfrak{E})$, $m(\mathfrak{E})$, $\delta(\mathfrak{E})$ the *algebra, monotone class, Dynkin class,* respectively, *generated by* $\mathfrak{E}$.

Remark 5. For $\gamma \in \{\sigma, \alpha, m, \delta\}$ and $\mathfrak{E}, \mathfrak{E}_1, \mathfrak{E}_2 \subset \mathfrak{P}(\Omega)$

(i) $\gamma(\mathfrak{E})$ is the smallest 'γ-class' that contains $\mathfrak{E}$,

(ii) $\mathfrak{E}_1 \subset \mathfrak{E}_2 \Rightarrow \gamma(\mathfrak{E}_1) \subset \gamma(\mathfrak{E}_2)$,

(iii) $\gamma(\gamma(\mathfrak{E})) = \gamma(\mathfrak{E})$.

Example 2. Let $\Omega = \mathbb{N}$ and $\mathfrak{E} = \{\{n\} : n \in \mathbb{N}\}$. Then

$$\alpha(\mathfrak{E}) = \{A \in \mathfrak{P}(\Omega) : A \text{ finite or } A^c \text{ finite}\} =: \mathfrak{A}.$$

Proof: $\mathfrak{A}$ is an algebra, see Example 1, and $\mathfrak{E} \subset \mathfrak{A}$. Thus $\alpha(\mathfrak{E}) \subset \mathfrak{A}$. On the other hand, for every finite set $A \subset \Omega$ we have $A = \bigcup_{n \in A} \{n\} \in \alpha(\mathfrak{E})$, and for every set $A \subset \Omega$ with finite complement we have $A = (A^c)^c \in \alpha(\mathfrak{E})$. Thus $\mathfrak{A} \subset \alpha(\mathfrak{E})$.

Moreover,

$$\sigma(\mathfrak{E}) = \mathfrak{P}(\mathbb{N}), \qquad m(\mathfrak{E}) = \mathfrak{E}, \qquad \delta(\mathfrak{E}) = \mathfrak{P}(\mathbb{N}).$$

Theorem 2. [Monotone class theorem, set version]

(i) $\mathfrak{E}$ closed w.r.t. intersections $\Rightarrow \sigma(\mathfrak{E}) = \delta(\mathfrak{E})$.

(ii) $\mathfrak{E}$ algebra $\Rightarrow \sigma(\mathfrak{E}) = m(\mathfrak{E})$.

Proof. Ad (i): Remark 3 implies

$$\delta(\mathfrak{E}) \subset \sigma(\mathfrak{E}).$$

We claim that

$$\delta(\mathfrak{E}) \text{ is closed w.r.t. intersections.} \tag{1}$$

Then, by Theorem 1.(ii),

$$\sigma(\mathfrak{E}) \subset \delta(\mathfrak{E}).$$

Put

$$\mathfrak{C}_B = \{C \subset \Omega : C \cap B \in \delta(\mathfrak{E})\}, \qquad B \in \delta(\mathfrak{E}),$$

so that (1) is equivalent to

$$\forall\, B \in \delta(\mathfrak{E}) : \delta(\mathfrak{E}) \subset \mathfrak{C}_B. \tag{2}$$

It is straightforward to verify that

$$\forall\, B \in \delta(\mathfrak{E}) : \mathfrak{C}_B \text{ Dynkin class.} \tag{3}$$

Moreover, since $\mathfrak{E}$ is closed w.r.t. intersections,

$$\forall\, E \in \mathfrak{E} : \mathfrak{E} \subset \mathfrak{C}_E.$$

Therefore

$$\forall\, E \in \mathfrak{E} : \delta(\mathfrak{E}) \subset \mathfrak{C}_E,$$

i.e., for all $E \in \mathfrak{E}, B \in \delta(\mathfrak{E})$, $E \cap B \in \delta(\mathfrak{E})$; hence

$$\forall\, B \in \delta(\mathfrak{E}) : \mathfrak{E} \subset \mathfrak{C}_B.$$

Since $\mathfrak{C}_B$ is a Dynkin system, $\delta(B) \subset \mathfrak{C}_B$.

Ad (ii): Obviously, $m(\mathfrak{E}) \subset \sigma(\mathfrak{E})$. By Part (ii) of Theorem 1, it is enough to show that $m(\mathfrak{E})$ is an algebra. This amounts to the claim that

$$m(\mathfrak{E}) \text{ is } {}^c\text{–closed and} \cap \text{–closed} . \tag{4}$$

First, the class

$$\mathfrak{C} := \{A \in m(\mathfrak{E}) \;:\; A^c \in m(\mathfrak{E})\}$$

is monotone, contains $\mathfrak{E}$ by assumption, and thus equals $m(\mathfrak{E})$. Second, in complete analogy to Part (i), for $B \in m(\mathfrak{E})$ it follows that the set

$$\mathfrak{C}_B = \{C \subset \Omega : C \cap B \in m(\mathfrak{E})\}$$

is a monotone class containing $\mathfrak{E}$ and thus $m(\mathfrak{E})$, so that $m(\mathfrak{E})$ is indeed $\cap$–closed. □

Lemma 1. $\mathfrak{E}$ semi-algebra $\Rightarrow \alpha(\mathfrak{E}) = \mathfrak{E}^+$.

Proof. Clearly $\mathfrak{E} \subset \mathfrak{E}^+ \subset \alpha(\mathfrak{E})$. It remains to show that $\mathfrak{E}^+$ is an algebra. For

$$A = \bigcup_{i=1}^{n} A_i \in \mathfrak{E}^+, \qquad A_i \in \mathfrak{E} \text{ disjoint,}$$

$$B = \bigcup_{i=1}^{n} B_i \in \mathfrak{E}^+, \qquad B_i \in \mathfrak{E} \text{ disjoint,}$$

$$A \cap B = \bigcup_{\substack{i \le n \\ j \le m}} (A_i \cap B_j), \qquad (A_i \cap B_j) \in \mathfrak{E} \text{ disjoint.}$$

Hence $\mathfrak{E}^+$ is $\cap$–stable. For

$$A = \bigcup_{i=1}^{n} A_i \in \mathfrak{E}^+, \qquad A_i \in \mathfrak{E} \text{ disjoint,}$$

with

$$A_i^c = \bigcup_{j \le n_i} B_j^i, \qquad B_j^i \in \mathfrak{E} \text{ disjoint,}$$

we have

$$\begin{aligned} A^c &= \bigcap_{i \le n} \bigcup_{j \le n_i} B_j^i \\ &= \bigcup_{\substack{(j_1,\ldots,j_n) \\ j_i \le n_i}} \Big(\underbrace{\bigcap_{i=1}^{n} B_{j_i}^i}_{\in \mathfrak{E} \text{ disjoint}} \Big). \end{aligned}$$

Hence $A^c \in \mathfrak{E}^+$, and $\mathfrak{E}^+$ is an algebra. □

Put

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\},$$

and equip this with the metric $d(x, y) := |\arctan(x) - \arctan(y)|$. Then $\overline{\mathbb{R}}$ is a complete, compact, separable, order complete metric space. For $a \in \mathfrak{R}$ set

$$(\pm\infty) + (\pm\infty) = a + (\pm\infty) = (\pm\infty) + a = \pm\infty, \qquad a/_{\pm\infty} = 0,$$

$$a \cdot (\pm\infty) = (\pm\infty) \cdot a = \begin{cases} \pm\infty & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ \mp\infty & \text{if } a < 0 \end{cases}$$

as well as $-\infty < a < \infty$.

Recall that $(\Omega, \mathfrak{G})$ is a *topological space* iff $\mathfrak{G} \subset \mathfrak{P}(\Omega)$ satisfies

(i) $\emptyset, \Omega \in \mathfrak{G}$,

(ii) $\mathfrak{G}$ is closed w.r.t. to intersections,

(iii) for every family $(G_i)_{i\in I}$ with $G_i \in \mathfrak{G}$ we have $\bigcup_{i\in I} G_i \in \mathfrak{G}$.

$\mathfrak{G}$ is the set of *open subsets* of Ω, and the complements of open sets are the *closed subsets* of Ω. $K \subset \Omega$ is *compact* iff for every family $(G_i)_{i\in I}$ with $G_i \in \mathfrak{G}$ and

$$K \subset \bigcup_{i\in I} G_i$$

there is a finite set $I_0 \subset I$ such that

$$K \subset \bigcup_{i\in I_0} G_i.$$

For $\Omega = \mathbb{R}^k$ and $\Omega = \overline{\mathbb{R}}^k$, we consider the natural (product) topologies $\mathfrak{G}_k$, $\overline{\mathfrak{G}}_k$.

Definition 4. For every topological space $(\Omega, \mathfrak{G})$

$$\mathfrak{B}(\Omega) = \sigma(\mathfrak{G})$$

is the *Borel-σ-algebra (in Ω w.r.t. $\mathfrak{G}$).* We shorten

$$\mathfrak{B} = \mathfrak{B}(\mathbb{R}), \quad \overline{\mathfrak{B}} = \mathfrak{B}(\overline{\mathbb{R}}), \quad \mathfrak{B}_k = \mathfrak{B}(\mathbb{R}^k), \overline{\mathfrak{B}}_k = \mathfrak{B}(\overline{\mathbb{R}}^k),$$

Remark 6. We have

$$\begin{aligned}\mathfrak{B}_k &= \sigma(\{F \subset \mathbb{R}^k : F \text{ closed}\}) = \sigma(\{K \subset \mathbb{R}^k : K \text{ compact}\}) \\ &= \sigma(\{]-\infty, a] : a \in \mathbb{R}^k\}) = \sigma(\{]-\infty, a] : a \in \mathbb{Q}^k\})\end{aligned}$$

and

$$\overline{\mathfrak{B}} = \{B \subset \overline{\mathbb{R}} : B \cap \mathbb{R} \in \mathfrak{B}\}. \tag{5}$$

One can prove that $\#\mathfrak{B}_k = \#\mathbb{R}^k$, and thus

$$\mathfrak{B}_k \subsetneq \mathfrak{P}(\mathbb{R}^k)$$

see Billingsley (1979, Exercise 2.21).

Definition 5. For any σ-algebra $\mathfrak{A}$ in Ω and $\widetilde{\Omega} \subset \Omega$

$$\widetilde{\mathfrak{A}} = \{\widetilde{\Omega} \cap A : A \in \mathfrak{A}\}$$

is the *trace-σ-algebra of* $\mathfrak{A}$ *in* $\widetilde{\Omega}$, sometimes denoted by $\widetilde{\Omega} \cap \mathfrak{A}$.

Remark 7.

(i) $\widetilde{\mathfrak{A}}$ is a σ-algebra in $\widetilde{\Omega}$.

(ii) $\widetilde{\mathfrak{A}} \not\subset \mathfrak{A}$ in general, but if $\widetilde{\Omega} \in \mathfrak{A}$, then $\widetilde{\mathfrak{A}} = \{A \in \mathfrak{A} : A \subset \widetilde{\Omega}\}$.

(iii) $\mathfrak{A} = \sigma(\mathfrak{E}) \Rightarrow \widetilde{\mathfrak{A}} = \sigma(\{\widetilde{\Omega} \cap E : E \in \mathfrak{E}\})$.

(iv) $\mathfrak{B}_k = \mathbb{R}^k \cap \overline{\mathfrak{B}}_k$, see (5) for $k = 1$.

(v) $[a, b[\cap \mathfrak{B}_k = \sigma(\{[a, c[: a \leq c \leq b\})$, see (iii).